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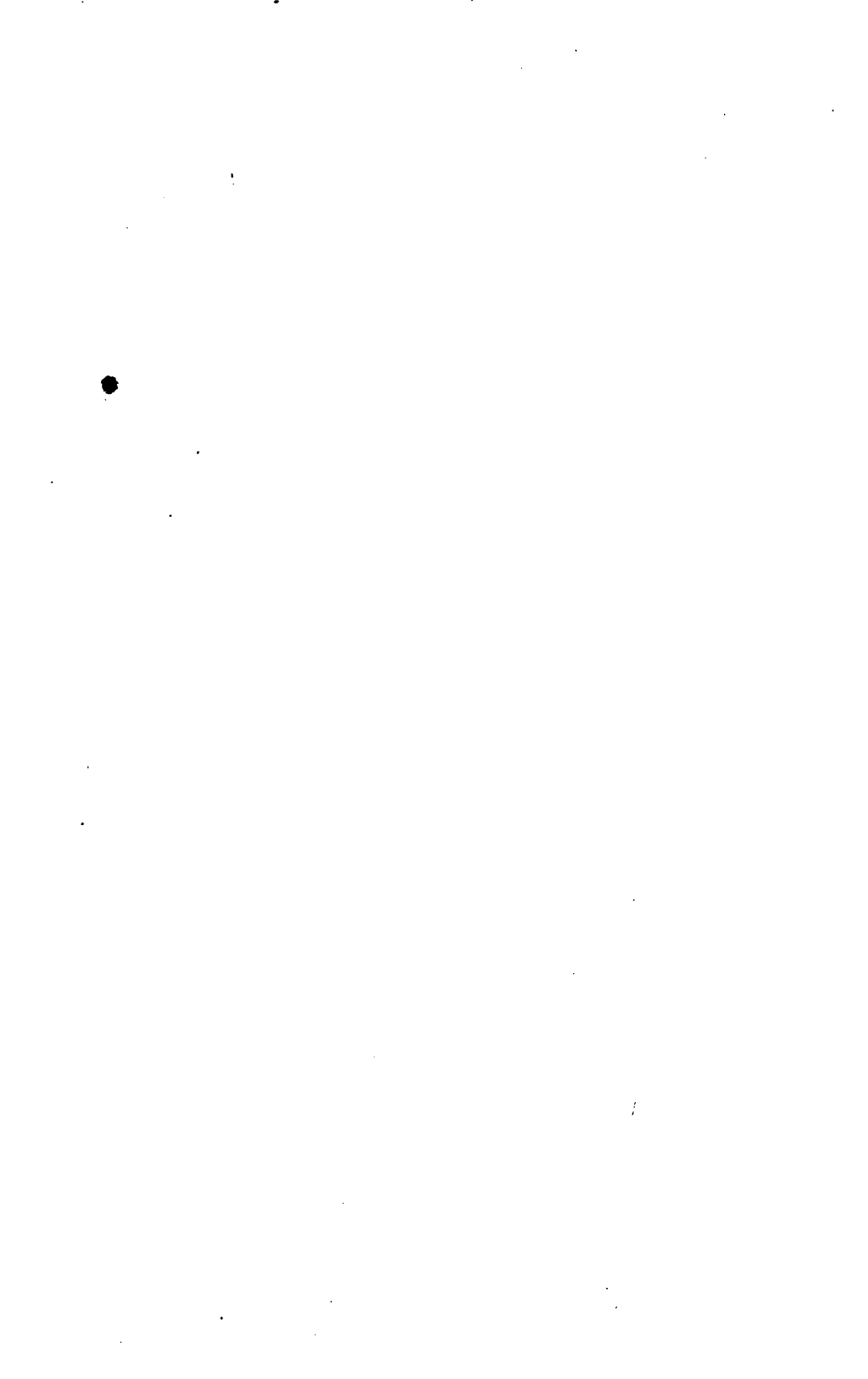
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FROM

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◦ ELEMENTS

OF

PLANE AND SPHERICAL

TRIGONOMETRY

BY

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GEOCKETT. PLANE AND SPHER. TRIGONOM.

W. P. I

PREFACE.

THIS work has been prepared for the use of beginners in the study of trigonometry. Assuming that a high degree of proficiency cannot be expected from such students, the author has limited himself to the selection of simple proofs of the formulas, not striving after original demonstrations. Geometrical proofs have been added in many cases, experience having shown that the student is assisted by them to a clearer understanding of the subject.

The student is expected, in technical institutions, to acquire facility in the use of the tables. All of the numerical examples have been computed by the author, with special attention to correctness in the last decimal place, and the arrangement of the computations has been carefully considered. Five-place tables have been adopted, and the angles in the examples are given to the nearest tenth of a minute, because the instruments ordinarily used by engineers are read by the vernier only to the nearest minute of arc, while the angle corresponding to a computed function may be found usually to the nearest tenth of a minute by the use of five-place tables.

Credit is due particularly to the works of Chauvenet, Snowball, Beasley, Woodhouse, Newcomb, and Todhunter, although many others have been consulted. A number of the illustrative examples in Art. 111 were taken from Gillespie's "Land Surveying," the numerical values being assigned by the author of that work.

The author cannot hope that among so many examples there are no errors; he therefore requests those finding such to kindly notify him.

GREEK ALPHABET.

A, α, α, α	<i>Alpha</i>	N, ν	<i>Nu</i>
B, β, β, β	<i>Beta</i>	Ξ, ξ	<i>Xi</i>
Γ, γ, γ, γ	<i>Gamma</i>	O, ο	<i>Omicron</i>
Δ, δ	<i>Delta</i>	Π, π	<i>Pi</i>
E, ε	<i>Epsilon</i>	P, ρ	<i>Rho</i>
Z, ζ	<i>Zeta</i>	Σ, σ, ς	<i>Sigma</i>
H, η	<i>Eta</i>	T, τ	<i>Tau</i>
Θ, θ	<i>Theta</i>	Υ, υ	<i>Upsilon</i>
I, ι	<i>Iota</i>	Φ, φ	<i>Phi</i>
K, κ	<i>Kappa</i>	X, χ	<i>Chi</i>
Λ, λ	<i>Lambda</i>	Ψ, ψ	<i>Psi</i>
M, μ	<i>Mu</i>	Ω, ω	<i>Omega</i>

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PART ONE.

PLANE AND ANALYTICAL TRIGONOMETRY.



CHAPTER I.

MEASUREMENT OF ANGLES; TRIGONOMETRIC FUNCTIONS OF ANGLES LESS THAN NINETY DEGREES.

1. **Analytical Trigonometry** treats of the relations of lines and angles by algebraic methods. In Plane and Spherical Trigonometry, these relations are applied to the solution of plane and spherical triangles.

2. **Directed Lines; Angles.** — A *directed line* is one whose beginning, direction, and length are known. The direction of the line is indicated by the order of the letters in its symbol; for instance, the line AB is drawn from A to B . If one direction along the line is considered positive, the opposite direction will be negative; thus, if the line AB is positive, the line BA will be negative, their numerical measures being equal, or

$$\text{line } AB = - \text{line } BA.$$

An *angle* is the figure formed by two intersecting lines, the point of intersection being the *vertex*.

The angle between any two given lines, whether intersecting or not intersecting,* is defined to be the same as the angle formed by two lines drawn through any point parallel to and in the same direction as the given lines. Hence an angle may be defined as the difference in direction of two directed lines.

* That is, parallel or in space.

3. Measurement of Angles.—Two methods of measuring angles are in common use,—the sexagesimal and the circular or natural methods.

4. Sexagesimal Measure.*—The circumference of a circle described about the vertex of the angle as a center is divided into 360 equal parts, and the angle at the center subtended by one of these parts is taken as the unit. The length of one of these divisions of the circle will depend upon its radius; but the corresponding angle at the center will be independent of the radius, since it is $\frac{1}{360}$ of four right angles. This unit angle, called a *degree*, is divided into 60 parts called *minutes*, each of which is subdivided into 60 parts called *seconds*. These are marked °, ', '' ; thus $43^{\circ} 14' 35''.2$ is read, "43 degrees, 14 minutes, and 35.2 seconds."

How many degrees are there in

- | | |
|---------------------------------------|-----------------------------|
| 1. Two thirds of four right angles ? | <i>Ans.</i> 240° . |
| 2. Two fifths of three right angles ? | <i>Ans.</i> 108° . |
| 3. Five sixths of two right angles ? | <i>Ans.</i> 150° . |

5. The Circular or Natural Measure.—From geometry we know that in any two concentric circles the arcs intercepted by any angle at the center are to each other as the radii of the circles. Therefore, if ACB be any central angle, we have

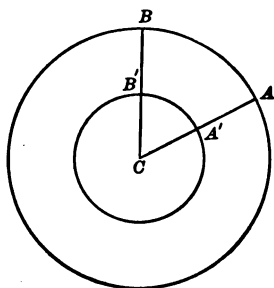


FIG. 1.

$$\frac{\text{arc } AB}{CA} = \frac{\text{arc } A'B'}{CA'}. \quad (1)$$

Hence the length of the intercepted arc divided by the radius is a number that is always the same for the same

angle, no matter what the radius may be.

We also know that in any circle any two central angles are to each other as their intercepted arcs, and therefore as the quotients of their intercepted arcs divided by the radius. We can, then, use these quotients to measure the angles.

* From *sexagesimus*, sixtieth.

The *circular measure of an angle* is the quotient obtained by dividing the length of its intercepted arc, in a circle whose center is at the vertex of the angle, by the radius of the circle. Thus, if c is the circular measure of the angle, l its intercepted arc, and r the radius, we have

$$c = \frac{l}{r}. \quad (2)$$

If the radius of the circle is unity,

$$c = l. \quad (3)$$

Hence the circular measure is represented by the length of the intercepted arc in the circle whose radius is unity.

The angle whose circular measure is one, that is, whose intercepted arc is equal to the radius, is called the *radian*.

1. The length of the intercepted arc of a central angle is 4 feet in a circle whose radius is 2 feet; the length of the intercepted arc of another central angle is 20 meters in a circle whose radius is 5 meters. Show that the second angle is twice as large as the first.

2. In a circle with a radius of 10 inches, the intercepted arc of a central angle is 5 inches, and that of an angle whose vertex is on the circumference is 10 inches. Find their circular measures. Ans. $\frac{1}{2}$.

6. Relation between the Two Measures. — Two right angles are measured by 180° , and also by $\pi r^* \div r = \pi$, since πr is the semicircumference of a circle whose radius is r . Hence, using the equality sign to represent “corresponds to,” we have

$$180^\circ = \pi \text{ in circular measure}; \quad (1)$$

$$\therefore 1^\circ = \frac{\pi}{180} \text{ in circular measure}. \quad (2)$$

$$\text{Again, } \pi \text{ in circular measure} = 180^\circ; \quad (3)$$

$$\therefore 1 \text{ in circular measure} = \frac{180^\circ}{\pi}. \quad (4)$$

$$\therefore 1 \text{ in circular measure} = 57^\circ.29577\ 95 + \quad (5)$$

$$\therefore 1 \text{ in circular measure} = 206\ 264'' .806. \quad (6)$$

$$1. \text{ What is the circular measure of } 120^\circ? \quad \text{Ans. } 120 \times \frac{\pi}{180} = \frac{2}{3} \pi.$$

$$2. \text{ What is the circular measure of } 10^\circ 10' 10''?$$

$$\begin{aligned} \text{The circular measure of } 1^\circ \text{ is } \frac{\pi}{180}, \text{ and that of } 1'' \text{ is } \frac{\pi}{180 \times 60 \times 60}. \text{ But} \\ 10^\circ 10' 10'' = 36610''. \quad \therefore \text{Circular measure of } 10^\circ 10' 10'' = \frac{36610 \pi}{180 \times 60 \times 60}. \end{aligned}$$

* π denotes the ratio of the circumference of a circle to its diameter, and is the number 3.14159 265+.

3. What is the sexagesimal measure of the angle whose circular measure is $\frac{1}{3}\pi$?

$$\pi = 180^\circ; \therefore \frac{1}{3}\pi = 60^\circ.$$

4. What is the sexagesimal measure of the angle whose circular measure is $\frac{2}{3}\pi$?

$$\text{Unity in circular measure} = \frac{180^\circ}{\pi}; \therefore \frac{2}{3} \text{ corresponds to } \frac{120^\circ}{\pi}.$$

5. What are the sexagesimal and circular measures corresponding to $\frac{2}{3}$ of three right angles?

$$\text{Ans. } 60^\circ; \frac{1}{3}\pi.$$

6. The sexagesimal measures of two angles are $22^\circ 30'$ and $43^\circ 14' 3''$.

Show that their circular measures are $\frac{1350\pi}{180 \times 60}$ and $\frac{155643\pi}{180 \times 60 \times 60}$.

7. The circular measures of three angles are $\frac{1}{2}\pi$, $\frac{2}{3}\pi$, and $\frac{1}{5}\pi$. Show that their sexagesimal measures are 15° , 40° , and $3^\circ 36'$.

8. The circular measures of three angles are $\frac{1}{4}$, $\frac{2}{3}$, and $\frac{1}{5}$. Show that their sexagesimal measures are $\frac{45^\circ}{\pi}$, $\frac{300^\circ}{\pi}$, and $\frac{40^\circ}{\pi}$.

9. Find the sexagesimal and circular measures corresponding to

(a) Seven tenths of four right angles. Ans. 252° ; $\frac{7}{5}\pi$.

(b) Five fourths of two right angles. Ans. 225° ; $\frac{5}{4}\pi$.

(c) Two thirds of one right angle. Ans. 60° ; $\frac{1}{3}\pi$.

7. Centesimal Measure. — In this system, proposed by the French, the right angle is divided into 100 parts called *grades*, each of which is subdivided into 100 parts called *minutes*, each minute being divided into 100 parts called *seconds*; marked $^\circ$, $'$, $''$.

8. Trigonometric Ratios. — Let the sides of a right-angled triangle be denoted as shown in Fig. 2. The trigonometric ratios may be defined as follows:

$$\left. \begin{array}{l} \text{The sine of an angle} = \frac{\text{side opposite}}{\text{hypotenuse}}; \text{ written } \sin A = \frac{o}{h} \\ \text{The cosine of an angle} = \frac{\text{side adjacent}}{\text{hypotenuse}}; \text{ written } \cos A = \frac{a}{h} \\ \text{The tangent of an angle} = \frac{\text{side opposite}}{\text{side adjacent}}; \text{ written } \tan A = \frac{o}{a} \\ \text{The cotangent of an angle} = \frac{\text{side adjacent}}{\text{side opposite}}; \text{ written } \cot A = \frac{a}{o} \\ \text{The secant of an angle} = \frac{\text{hypotenuse}}{\text{side adjacent}}; \text{ written } \sec A = \frac{h}{a} \\ \text{The cosecant of an angle} = \frac{\text{hypotenuse}}{\text{side opposite}}; \text{ written } \operatorname{cosec} A = \frac{h}{o} \end{array} \right\} (1)$$

These fundamental equations should be thoroughly memorized.

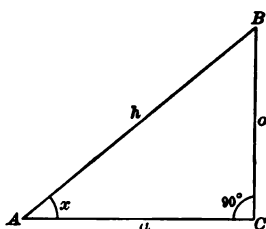


FIG. 2.

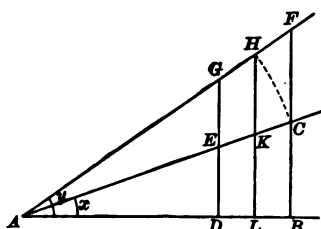


FIG. 3.

9. **The Ratios are Constant for Any One Angle.** — In Fig. 3 let BAC and BAF be two angles differing by a quantity as small as we please. At any two points B and D on AB , draw BF and DG perpendicular to AF ; with A as a center, and radius AC , describe the arc CH , and draw LH perpendicular to AB . The triangles BAC and DAE are similar.

$$\therefore \frac{BC}{AC} = \frac{DE}{AE} = \text{a constant} = \frac{\text{side opposite}}{\text{hypotenuse}} = \sin x.$$

$$\frac{BC}{AB} = \frac{DE}{AD} = \text{a constant} = \frac{\text{side opposite}}{\text{side adjacent}} = \tan x.$$

$$\frac{AC}{AB} = \frac{AE}{AD} = \text{a constant} = \frac{\text{hypotenuse}}{\text{side adjacent}} = \sec x.$$

10. **The Values of the Ratios differ for Different Angles.** — From Fig. 3 we have, since $AH = AC$,

$$\sin x = \frac{BC}{AC} \text{ and } \sin y = \frac{LH}{AH} = \frac{LH}{AC};$$

$$\tan x = \frac{BC}{AB} \text{ and } \tan y = \frac{BF}{AB};$$

$$\sec x = \frac{AC}{AB} \text{ and } \sec y = \frac{AF}{AB}.$$

11. **The Angle may be constructed when One of the Ratios is known.** — Let $\sin x = \frac{1}{2}$. With any convenient radius AC , describe a circle about A as a center. Draw AD perpendicular to AB , and on it lay off $AD = \frac{1}{2} AC$; draw DC parallel to AB until

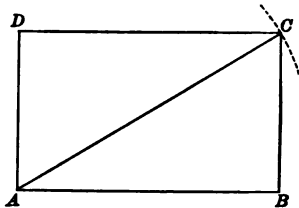


FIG. 4.

it intersects the circle at C ; join A and C , and BAC will be the required angle, since

$$\sin BAC = \frac{BC}{AC} = \frac{AD}{AC} = \frac{1}{2}.$$

Let $\tan x = \frac{3}{4}$. Lay off any convenient distance AB ; at B draw BC perpendicular to AB , and lay off $BC = \frac{3}{4} AB$; join A and C , and BAC will be the required angle, since

$$\tan BAC = \frac{BC}{AB} = \frac{3}{4}.$$

Let $\sec x = 2$. Lay off any convenient distance AB ; erect the perpendicular line BC ; with a radius $AC = 2 AB$ describe an arc cutting BC at C ; join A and C , and BAC will be the required angle, since

$$\sec BAC = \frac{AC}{AB} = 2.$$

Let the student construct the angle whose cosine is $\frac{1}{2}$, the angle whose co-tangent is 5, and the angle whose cosecant is 4.

12. We therefore conclude that to any one angle there will correspond a special value of each of these ratios, that the value of each ratio will differ for different angles, and that, if any one of these ratios is given, the angle may be constructed.

13. Tables of Sines, Cosines, etc. — The values of these ratios for angles between 0° and 90° have been computed, and are given in tables so arranged that the values corresponding to any angle may be readily found. The tables of *natural sines*, etc., contain the actual values of these ratios; while the tables of *logarithmic sines*, etc., contain their logarithms.

14. Ratios for 30° , 45° , 60° .

(a) *Ratios for 45° .* — In Fig. 5 let the angle $A = 45^\circ$; then $B = 90^\circ - A = 45^\circ$.

$\therefore AC = CB$, since they are opposite equal angles.

Let $AC = a$; then $CB = a$, and $AB = \sqrt{a^2 + a^2} = a\sqrt{2}$.

$$\therefore \sin 45^\circ = \frac{CB}{AB} = \frac{1}{\sqrt{2}}; \quad \tan 45^\circ = \frac{CB}{AC} = 1; \quad \sec 45^\circ = \frac{AB}{AC} = \sqrt{2};$$

$$\cos 45^\circ = \frac{AC}{AB} = \frac{1}{\sqrt{2}}; \quad \cot 45^\circ = \frac{AC}{CB} = 1; \quad \operatorname{cosec} 45^\circ = \frac{AB}{CB} = \sqrt{2}.$$

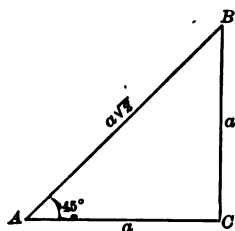


FIG. 5.

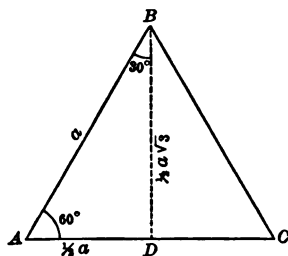


FIG. 6.

(b) *Ratios for 30° and 60°.*—In the equilateral triangle ABC (Fig. 6), let $AB = a$; draw DB perpendicular to AC ; AC will be bisected at D , making $AD = \frac{1}{2}a$, and the angle $ABD = \text{angle } DBC = 30^\circ$.

$$\text{Also } DB = \sqrt{a^2 - \frac{1}{4}a^2} = \frac{1}{2}a\sqrt{3}.$$

$$\therefore \sin ABD = \sin 30^\circ = \frac{AD}{AB} = \frac{1}{2}; \quad \tan 30^\circ = \frac{AD}{DB} = \frac{1}{\sqrt{3}}; \quad \sec 30^\circ = \frac{AB}{DB} = \frac{2}{\sqrt{3}};$$

$$\cos 30^\circ = \frac{DB}{AB} = \frac{\sqrt{3}}{2}; \quad \cot 30^\circ = \frac{DB}{AD} = \sqrt{3}; \quad \operatorname{cosec} 30^\circ = \frac{AB}{AD} = 2.$$

$$\sin DAB = \sin 60^\circ = \frac{DB}{AB} = \frac{\sqrt{3}}{2}; \quad \tan 60^\circ = \frac{DB}{AD} = \sqrt{3}; \quad \sec 60^\circ = \frac{AB}{AD} = 2;$$

$$\cos 60^\circ = \frac{AD}{AB} = \frac{1}{2}; \quad \cot 60^\circ = \frac{AD}{DB} = \frac{1}{\sqrt{3}}; \quad \operatorname{cosec} 60^\circ = \frac{AB}{DB} = \frac{2}{\sqrt{3}}.$$

Note that the sines of 30° , 45° , and 60° , are $\frac{1}{2}\sqrt{1}$, $\frac{1}{2}\sqrt{2}$, and $\frac{1}{2}\sqrt{3}$ respectively.

15. The Ratios are not Independent of Each Other; for we have from Fig. 2,

$$h^2 = a^2 + c^2,$$

so that if two of the three quantities h , c , and a , are given, the third can be found. Hence if we know one of the ratios, that is, the relative values of two of the three elements, we can determine the relative value of the third element, and from it the other ratios.

Thus if $\tan x = \frac{3}{4}$, and the other ratios are required, we have

$$\tan x = \frac{o}{a} = \frac{3}{4}; \text{ let } o = 3, a = 4; \text{ then } h = 5.$$

$$\therefore \sin x = \frac{o}{h} = \frac{3}{5}; \cos x = \frac{a}{h} = \frac{4}{5}; \cot x = \frac{a}{o} = \frac{4}{3};$$

$$\sec x = \frac{h}{a} = \frac{5}{4}; \operatorname{cosec} x = \frac{h}{o} = \frac{5}{3}.$$

Having given the ratio on the left, find the ratios on the right:

	$\sin x.$	$\cos x.$	$\tan x.$	$\cot x.$	$\sec x.$	$\operatorname{cosec} x.$
1. $\sin x = \frac{8}{17}$	—	$\frac{15}{17}$	$\frac{8}{15}$	$\frac{15}{8}$	$\frac{17}{15}$	$\frac{17}{8}$
2. $\cos x = \frac{5}{13}$	$\frac{12}{13}$	—	$\frac{12}{5}$	$\frac{5}{12}$	$\frac{13}{5}$	$\frac{13}{12}$
3. $\tan x = \frac{7}{24}$	$\frac{7}{25}$	$\frac{24}{25}$	—	$\frac{24}{7}$	$\frac{25}{24}$	$\frac{25}{7}$
4. $\cot x = 2$	$\frac{1}{\sqrt{5}}$	$\frac{2}{\sqrt{5}}$	$\frac{1}{2}$	—	$\frac{1}{2}\sqrt{5}$	$\sqrt{5}$
5. $\sec x = \frac{29}{20}$	$\frac{21}{29}$	$\frac{20}{29}$	$\frac{21}{20}$	$\frac{20}{21}$	—	$\frac{29}{21}$
6. $\operatorname{cosec} x = 3$	$\frac{1}{3}$	$\frac{2}{3}\sqrt{2}$	$\frac{1}{4}\sqrt{2}$	$2\sqrt{2}$	$\frac{3}{4}\sqrt{2}$	—

16. Measurement of Angles in the Field.—In Fig. 7, $FGHK$ represents a fixed graduated circle, and $ABDE$ a circle resting on the plate $FGHK$, and capable of moving about a pivot at C ; I and O are two small rods fixed to $ABDE$, and perpendicular to the planes of the circles; and M is a mark on the circle $ABDE$ in the same line with I , C , and O . If we wish to measure the horizontal angle between two distant objects, two church towers, for example, we proceed as follows: first place the circles in a horizontal position; revolve the circle $ABDE$, looking along the line IO , until the line of

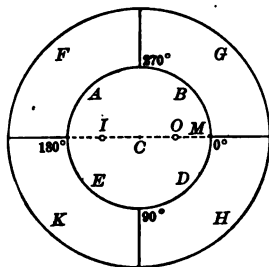


FIG. 7.

sight passes through one of the objects, and note the reading

of the circle opposite the mark *M*; then revolve the circle *ABDE*, being careful not to move *FGHK*, until the line of sight passes through the second object, and note the new reading of the circle opposite the mark *M*. The difference between the two readings will be the angular distance required.

17. The Engineers' Transit, shown in Fig. 8, is used in measuring horizontal and vertical angles. The lower circle is provided with two levels, by which its horizontality is tested.

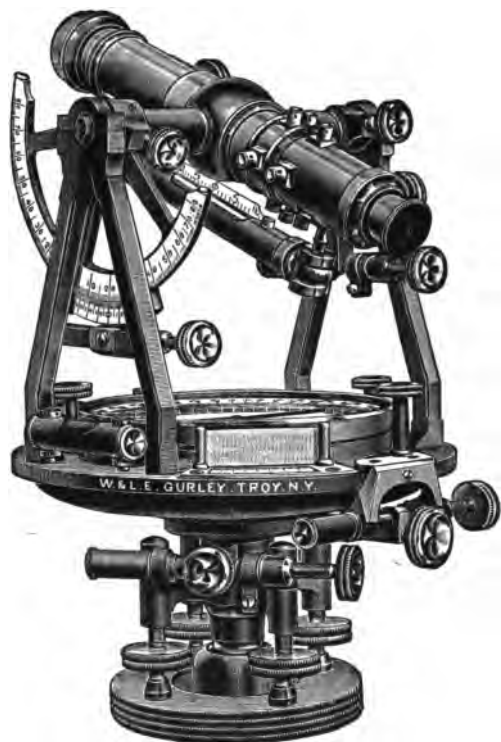


FIG. 8.

The rods *I* and *O* are replaced by the telescope with a system of intersecting wires in the common focus of the object glass and eyepiece, the telescope being capable of rotation about an axis parallel to the horizontal circle. The circle fixed to the axis of the telescope is vertical when the plate bearing the upright supports is horizontal.

18. Illustrations of the Application of the Ratios.*

1. A rope fastened to the top of a vertical pole 60 feet high, and to a stake driven in the ground, is inclined at an angle of 30° . How far is the stake from the bottom of the pole? How long is the rope?

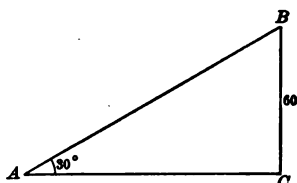


FIG. 9.

$$\frac{CB}{AC} = \tan 30^\circ = \frac{1}{\sqrt{3}}.$$

$$\therefore AC = \sqrt{3} CB = 60\sqrt{3} \text{ feet.}$$

$$\frac{CB}{AB} = \sin 30^\circ = \frac{1}{2}.$$

$$\therefore AB = 2 CB = 120 \text{ feet.}$$

2. The angle at the vertex of a right circular cone is 60° , and the slant height is 10 inches. What is the altitude and the radius of the base of the cone?

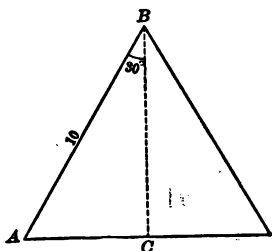


FIG. 10.

$$\frac{CB}{AB} = \cos 30^\circ = \frac{\sqrt{3}}{2}.$$

$$\therefore CB = \frac{\sqrt{3}}{2} AB = 5\sqrt{3} \text{ inches.}$$

$$\frac{AC}{AB} = \sin 30^\circ = \frac{1}{2}.$$

$$\therefore AC = \frac{1}{2} AB = 5 \text{ inches.}$$

3. The top of a ladder 30 feet long rests on the upper edge of a wall 15 feet high. What is the inclination of the ladder?

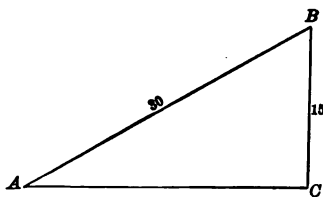


FIG. 11.

$$\sin CAB = \frac{CB}{AB} = \frac{15}{30} = \frac{1}{2};$$

$$\text{but } \sin 30^\circ = \frac{1}{2}. \therefore CAB = 30^\circ.$$

In these cases the ratios corresponding to the angles were known from Art. 14. Usually it will be necessary to refer to the tables in solving problems involving the ratios.

* It is assumed that the ground is horizontal.

CHAPTER II.

RIGHT PLANE TRIANGLES.

19. It has been shown in Geometry that a right-angled triangle can be constructed when two elements* besides the right angle are known, one of the known elements being a side. We also know that

(1) The hypotenuse is greater than either of the other two sides.

(2) The hypotenuse is less than the sum of the other two sides.

(3) The sum of the two acute angles must be 90° .

(4) The greater side is opposite the greater angle.

(5) The square on the hypotenuse is equal to the sum of the squares on the other two sides.

20. A triangle is said to be *solved* when, having some of the elements given, the others have been found by some process.

21. The Solution of a Right Triangle is effected by means of the trigonometric ratios. Each equation,

as $\sin A = \frac{a}{c}$, contains three

quantities; and two of them must be known in order that the third may be found. Hence in any particular case we use the equations that contain the two given elements; thus, if

a and b are given, we use $\tan A = \frac{a}{b}$

to find A , and then c may be found from either $\sin A = \frac{a}{c}$ or

$\cos A = \frac{b}{c}$.

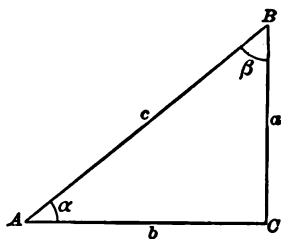
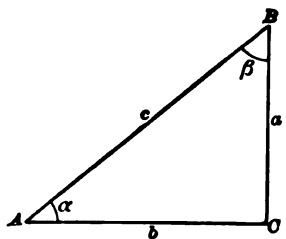


FIG. 12.

* The elements of a triangle are the three sides and the three angles.

The equations used in the solution of right triangles are



$$\left. \begin{aligned} \sin A &= \frac{a}{c} = \cos B. \\ \cos A &= \frac{b}{c} = \sin B. \\ \tan A &= \frac{a}{b} = \cot B. \\ \cot A &= \frac{b}{a} = \tan B. \\ A + B &= 90^\circ. \\ c^2 &= a^2 + b^2. \end{aligned} \right\} \quad (1)$$

FIG. 13.

22. From the Trigonometric Ratios we have

$$\left. \begin{aligned} \tan A &= \frac{a}{b}; \quad \therefore a = b \tan A, \\ \cot B &= \frac{a}{b}; \quad \therefore a = b \cot B, \end{aligned} \right\} \quad (1)$$

or, *any side of a right triangle is equal to the other side multiplied by the tangent of the angle opposite, or by the cotangent of the angle adjacent, to the side itself.*

$$\left. \begin{aligned} \sin A &= \frac{a}{c}; \quad \therefore a = c \sin A, \\ \cos B &= \frac{a}{c}; \quad \therefore a = c \cos B, \end{aligned} \right\} \quad (2)$$

or, *any side is equal to the hypotenuse multiplied by the sine of the opposite angle, or by the cosine of the adjacent angle.*

$$\left. \begin{aligned} \sec A &= \frac{c}{b}; \quad \therefore c = b \sec A, \\ \operatorname{cosec} B &= \frac{c}{b}; \quad \therefore c = b \operatorname{cosec} B, \end{aligned} \right\} \quad (3)$$

or, *the hypotenuse is equal to a side multiplied by the secant of the adjacent angle, or by the cosecant of the opposite angle.*

NOTE. — The secant of an angle is the reciprocal of its cosine, and the cosecant is the reciprocal of its sine; hence the logarithm of the secant is the arithmetical complement of that of the cosine, and the logarithm of the cosecant is the A. C. of that of the sine, or

$$\log \sec x = \operatorname{colog} \cos x, \text{ and } \log \operatorname{cosec} x = \operatorname{colog} \sin x.$$

23. Case I. Given c and A .

$$\text{Formulas: } \begin{cases} a = c \sin A. \\ b = c \cos A. \\ B = 90^\circ - A. \end{cases}$$

 1. Solve the triangle when $c = 1.0034$, and $A = 42^\circ 10' .3$.

$$\therefore B = 90^\circ - A = 47^\circ 49' .7.$$

(a) By natural functions.

$$a = c \sin A = 1.0034 \times 0.67136 = 0.67364.$$

$$b = c \cos A = 1.0034 \times 0.74114 = 0.74366.$$

(b) By the use of logarithms.

$$a = c \sin A; \therefore \log a = \log c + \log \sin A.$$

$$b = c \cos A; \therefore \log b = \log c + \log \cos A.$$

Always write first all the formulas that will be used in the problem; then write them in a form adapted to logarithmic computation; then refer to the tables and write the logarithms in their proper places. Thus in this case we arrange the work as follows:

$$\left. \begin{array}{ll} \log c = & \log c = \\ + \log \sin A = & + \log \cos A = \\ \therefore \log a = & \therefore \log b = \\ \therefore a = & \therefore b = \end{array} \right\} \text{ or } \left\{ \begin{array}{ll} + \log \sin A = & (1) \\ \log c = & (3) \\ + \log \cos A = & (2) \\ \therefore \log a = (1) + (3) & (4) \\ a = & (6) \\ \therefore \log b = (2) + (3) & (5) \\ b = & (7) \end{array} \right.$$

The positive signs preceding $\log \sin A$ and $\log \cos A$ indicate that they are to be added to $\log c$.

We now find the angle A in the table of logarithmic functions and take from the table both $\log \sin A$ and $\log \cos A$, writing them in their proper places. Then we refer to the table of logarithms of numbers and find $\log c$, writing it opposite $\log c$. Then we add the proper quantities to find $\log a$ and $\log b$, finally looking in the table of the logarithms of numbers for the numbers corresponding to the computed values of $\log a$ and $\log b$.

The arrangement on the right is preferable, since it saves

the writing of one line. The numbers in the parentheses indicate the order in which the quantities should be found.

$\begin{array}{r} \log c = 0.00147 \\ + \log \sin A = 9.82695 - 10 \\ \hline \log a = 9.82842 - 10 \\ a = 0.67363 \end{array}$	$\begin{array}{r} \log c = 0.00147 \\ + \log \cos A = 9.86990 - 10 \\ \hline \log b = 9.87137 - 10 \\ b = 0.74365 \end{array}$	$\begin{array}{r} \text{or } \log \sin A = 9.82695 - 10 \\ \log c = 0.00147 \\ + \log \cos A = 9.86990 - 10 \\ \hline \log a = 9.82842 - 10 \\ a = 0.67363 \\ \log b = 9.87137 - 10 \\ b = 0.74365 \end{array}$
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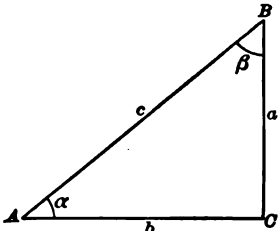


FIG. 14.

Check :

$$\begin{aligned} a^2 &= (c + b)(c - b) \\ c + b &= 1.74705 \\ c - b &= 0.25975 \\ \hline \log(c + b) &= 0.24230 \\ \log(c - b) &= 9.41456 - 10 \\ \therefore \log a^2 &= 9.65686 - 10 \\ \log a &= 9.82843 - 10 \end{aligned}$$

Exact agreement is not expected, since the tables give the values of the functions only to the *nearest* unit in the fifth decimal place. The -10 is usually omitted, and $\sin A$ is written for $\log \sin A$, when there is no danger of confusion.

2. Solve the triangle when $c = 34.687$, and $B = 49^\circ 8'4$.

$$\text{Ans. } A = 40^\circ 51'.6; b = 26.234; a = 22.6925.$$

3. Solve the triangle when $c = 305$, and $A = 63^\circ 31'.14$, using the natural functions.

$$\text{Ans. } a = 273.00; b = 136.00.$$

4. Solve the triangle when $c = 205$, and $B = 49^\circ 33'.01$, using the natural functions.

$$\text{Ans. } a = 133.00; b = 156.00.$$

24. Case II. Given c and a .

$$\text{Formulas : } \begin{cases} \sin A = \frac{a}{c} \\ b = a \cot A = c \cos A \\ B = 90^\circ - A \end{cases}$$

1. Solve the triangle when $c = 8.7982$, and $a = 3.1292$.

$$\therefore \log \sin A = \log a - \log c; \log b = \log a + \log \cot A = \log c + \log \cos A.$$

$\begin{array}{r} \log a = 0.49544 \\ - \log c = 0.94439 \\ \hline \log \sin A = 9.55105 - 10 \\ A = 20^\circ 50'.1 \\ B = 69^\circ 9'.9 \end{array}$	$\begin{array}{r} \log a = 0.49544 \\ + \log \cot A = 0.41958 \\ \hline \log b = 0.91502 \\ b = 8.2228 \end{array}$	$\begin{array}{r} \log c = 0.94439 \\ + \log \cos A = 9.97063 - 10 \\ \hline \log b = 0.91502 \\ b = 8.2228 \end{array}$
---	---	--

or * $\log \cot A = 0.41958$ (5) Check: $b^2 = (c - a)(c + a)$
 $\log a = 0.49544$ (1) $c - a = 5.6690$, $\log(c - a) = 0.75351$
 $-\log c = 0.94439$ (2) $c + a = 11.9274$, $\log(c + a) = 1.07655$
 $\log \cos A = 9.97063$ (6) $\log b^2 = 1.83006$
 $\log \sin A = 9.55105$ (1) - (2) $\log b = 0.91503$
 $A = 20^\circ 50' 1$ (4)
 $B = 69^\circ 9' 9$
 $\log b = 0.91502$ { (1) + (5)
 $b = 8.2228$ { (2) + (6)

2. Solve the triangle when $c = 369.27$, and $b = 235.64$.

Ans. $A = 50^\circ 20' 9$; $B = 39^\circ 39' 1$; $a = 284.31$.

3. Solve the triangle when $c = 281$, and $a = 180$, using the natural functions.

Ans. $A = 34^\circ 42' 5$; $b = 231.00$ or 231.01 .

4. Solve the triangle when $c = 365$, and $b = 76$, using the natural functions.

Ans. $A = 77^\circ 58' 93$; $a = 357.00$.

25. Case III. Given a and b .

Formulas:
$$\begin{cases} \tan A = \frac{a}{b} \\ c = \frac{a}{\sin A} = \frac{b}{\cos A} \\ B = 90^\circ - A. \end{cases}$$

1. Solve the triangle when $a = 169.03$, and $b = 203.44$.

$\therefore \log \tan A = \log a - \log b$; $\log c = \log a - \log \sin A = \log b - \log \cos A$.

$\log a = 2.22796$	$\log a = 2.22796$	$\log b = 2.30843$
$-\log b = 2.30843$	$-\log \sin A = 9.80555 - 10$	$-\log \cos A = 9.88602 - 10$
$\log \tan A = 9.91953 - 10$	$\log c = 2.42241$	$\log c = 2.42241$
$A = 39^\circ 43' 3$	$c = 264.49$	$c = 264.49$
$B = 50^\circ 16' 7$		

or * $\log a = 2.22796$ (1) Check: $a^2 = c^2 - b^2$
 $\log \sin A = 9.80555$ (5) $c + b = 467.93$
 $\log \cos A = 9.88602$ (6) $c - b = 61.05$
 $\log b = 2.30843$ (2) $\log(c + b) = 2.67018$
 $\therefore \log \tan A = 9.91953$ (3) $\log(c - b) = 1.78569$
 $A = 39^\circ 43' 3$ (4) $\therefore \log a^2 = 4.45587$
 $B = 50^\circ 16' 7$ $\log a = 2.22794$
 $\log c = 2.42241$ { (1) - (5)
 $c = 264.49$ { (2) - (6)

* This form is preferable.

2. Solve the triangle when $a = 4.8199$, and $b = 2.6492$.

$$\text{Ans. } A = 61^\circ 12'.3; B = 28^\circ 47'.7; c = 5.4999.$$

3. Solve the triangle when $a = 60$, and $b = 91$, using the natural functions.

$$\text{Ans. } A = 33^\circ 23'.9; c = 109.00.$$

4. Solve the triangle when $a = 72$, and $b = 65$, using the natural functions.

$$\text{Ans. } A = 47^\circ 55'.5; c = 97.000.$$

26. Case IV. Given a and A .

$$\text{Formulas: } \begin{cases} b = a \cot A. \\ c = \frac{a}{\sin A} = \frac{b}{\cos A}. \\ B = 90^\circ - A. \end{cases}$$

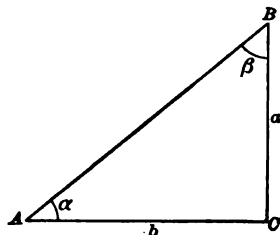


FIG. 15.

1. Solve the triangle when $a = 613.35$, and $A = 40^\circ 12'.6$.

$$\therefore B = 90^\circ - A = 49^\circ 47'.4.$$

$$\log b = \log a + \log \cot A.$$

$$\log c = \log a - \log \sin A = \log b - \log \cos A.$$

$\log a = 2.78770$	$\log a = 2.78770$	$\log b = 2.86065$
$+ \log \cot A = 0.07295$	$- \log \sin A = 9.80996 - 10$	$- \log \cos A = 9.88291 - 10$
$\log b = 2.86065$	$\log c = 2.97774$	$\log c = 2.97774$
$b = 725.52$	$c = 950.04$	$c = 950.04$

or $\log \sin A = 9.80996$ (1)

$\log a = 2.78770$ (3)

$\log \cot A = 0.07295$ (2)

$\log c = 2.97774$ (3) - (1)

$c = 950.04$

$\log b = 2.86065$ (3) + (2)

$b = 725.52$

Check: $a^2 = (c + b)(c - b)$

$c + b = 1675.56, \log(c + b) = 3.22416$

$c - b = 224.52, \log(c - b) = 2.35126$

$\log a^2 = 5.57542$

$\log a = 2.78771$

2. Solve the triangle when $a = 3.6378$, and $B = 69^\circ 23'.5$.

$$\text{Ans. } A = 20^\circ 36'.5; b = 9.6738; c = 10.335.$$

3. Solve the triangle when $b = 160$, and $A = 55^\circ 17'.48$, using the natural functions.

$$\text{Ans. } c = 281.00; a = 231.00.$$

4. Solve the triangle when $a = 340$, and $A = 60^\circ 55'.85$, using the natural functions.

$$\text{Ans. } c = 389.00; b = 189.00.$$

27. Isosceles Triangles. — If a perpendicular to the base is drawn from the vertex, it will bisect the base and the angle at the vertex, forming two equal right triangles.

$$\angle ABD = \angle DBC = \frac{1}{2} \beta; \quad AB = BC;$$

$$AD = DC = \frac{1}{2} b.$$

1. Solve the triangle when $b = 2.1452$, and $\beta = 121^\circ 14' .6$.

$$\therefore AD = 1.0726; \quad ABD = 60^\circ 37' .3;$$

$$\alpha = 90^\circ - \frac{1}{2} \beta = 29^\circ 22' .7.$$

$$a = \frac{\frac{1}{2} b}{\sin \frac{1}{2} \beta}; \quad \therefore \log a = \log \frac{1}{2} b - \log \sin \frac{1}{2} \beta.$$

$$p = \frac{1}{2} b \cot \frac{1}{2} \beta; \quad \therefore \log p = \log \frac{1}{2} b + \log \cot \frac{1}{2} \beta.$$

$$\log \frac{1}{2} b = 0.03044$$

$$\log \frac{1}{2} b = 0.03044$$

$$- \log \sin \frac{1}{2} \beta = 9.94022 - 10$$

$$+ \log \cot \frac{1}{2} \beta = 9.75049 - 10$$

$$\log a = 0.00022$$

$$\log p = 9.78093 - 10$$

$$a = 1.2309$$

$$p = 0.60385$$

2. Solve the triangle when $\alpha = 52^\circ 10' .2$, and $a = 600.2$.

$$\text{Ans. } \beta = 75^\circ 39' .6; \quad \frac{1}{2} b = 368.12; \quad p = 474.07.$$

28. Given c and b (Special Method). — When b nearly equals c , the angle found from the formula $\cos A = \frac{b}{c}$ is uncertain, the tabular difference for the cosine being so small that a small error in $\cos A$ would produce a large error in A .

In the figure, AD bisects the angle A , and DE is perpendicular to AB ; $\therefore DE = CD$. Let $CD = x = DE$;

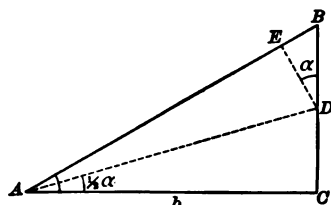


FIG. 17.

$$\therefore \tan \frac{1}{2} \alpha = \frac{x}{b} \quad (1)$$

$$\text{Also,} \quad CB = a = CD + DB = CD + DE \sec \alpha;$$

$$\therefore a = x + x \sec \alpha; \quad \therefore x = \frac{a}{1 + \sec \alpha};$$

$$\therefore x = \frac{a}{1 + \frac{c}{b}} = \frac{ab}{c + b}. \quad (2)$$

From (1) and (2),

$$\tan \frac{1}{2} \alpha = \frac{a}{c+b} = \frac{\sqrt{c^2 - b^2}}{c+b} = \sqrt{\frac{(c+b)(c-b)}{(c+b)^2}};$$

$$\therefore \tan \frac{1}{2} \alpha = \sqrt{\frac{c-b}{c+b}}. \quad (3)$$

Suppose that we wish to find the greatest distance at sea at which a mountain 4.3 miles high can be seen, the earth being considered as a sphere with a radius of 3963.3 miles, and the distance being measured as a chord.

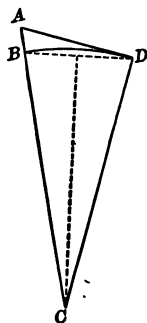


FIG. 18.

Let $BA=4.3$, and $CB=CD=3963.3$; BD being the distance required. Then $\cos DCA = \frac{CD}{CA}$, giving $\log \cos DCA = 9.99952$; and DCA as found from the tables might have any value between $2^\circ 40' 5$ and $2^\circ 42' 5$.

Using (3), we have

$$CA - CD = 4.3; \log = 0.63347$$

$$CA + CD = 7930.9; \log = 3.89932$$

$$2) 6.73415 - 10$$

$$\log \tan \frac{1}{2} DCA = 8.36708 - 10$$

$$\text{Cpl. } T' = 3.53620$$

$$\log (\frac{1}{2} DCA)' = 1.90328$$

$$\therefore \frac{1}{2} DCA = 80'.035; \therefore DCA = 2^\circ 40'.07.$$

Then $BD = 2CD \sin \frac{1}{2} DCA$ will give the chord BD . The arc BD is found from the proportion:

$$360^\circ : DCA = 2\pi \times 3963.3 : \text{arc } BD.$$

NOTE. — Eq. (3) follows directly from (4), Art. 69:

$$\tan \frac{1}{2} \alpha = \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}, \text{ where } \cos \alpha = \frac{b}{c}.$$

29. Given α and b (Special Method). —

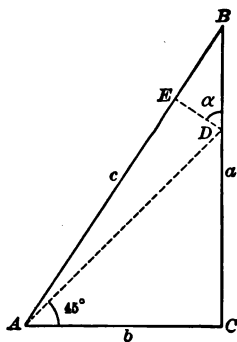
When α and b are nearly equal, the angle α may be determined more accurately, as follows:

Draw AD , making $CAD = 45^\circ$, and DE perpendicular to AB . Then

$$\tan DAE = \tan(\alpha - 45^\circ) = \frac{DE}{AE}.$$

$$\text{But } DE = DB \cos \alpha = (CB - CD) \cos \alpha$$

$$= (a - b) \cos \alpha = \frac{(a - b)b}{c},$$



and

$$\begin{aligned}
 AE &= AB - EB = AB - DB \sin \alpha = c - \frac{(a-b)a}{c} = \frac{c^2 - a^2 + ab}{c} \\
 &= \frac{b^2 + ab}{c}. \\
 \therefore \frac{DE}{AE} &= \frac{(a-b)b}{ab + b^2} = \frac{a-b}{a+b}. \\
 \therefore \tan(\alpha - 45^\circ) &= \frac{a-b}{a+b}. \quad (1)
 \end{aligned}$$

If b were greater than a , the formula would be

$$\tan(45^\circ - \alpha) = \frac{b-a}{b+a}. \quad (2)$$

NOTE.—Eq. (1) may be found from the relation proved in Art. 100 :

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(\alpha - \beta)}{\tan \frac{1}{2}(\alpha + \beta)}, \text{ where } \frac{1}{2}(\alpha + \beta) = 45^\circ, \text{ and } \frac{1}{2}(\alpha - \beta) = \alpha - 45^\circ.$$

EXAMPLES.

NOTE.—The angle between the line of sight and a horizontal plane is called an *angle of elevation* when the point sighted on is above the horizontal plane, and an *angle of depression* when it is below the horizontal plane.

1. The shadow of a vertical pole 30 feet high is 40 feet long. Find the elevation of the sun above the horizon. Ans. $36^\circ 52'2$.

2. The vertical central pole of a circular tent is 20 feet high, and its top is fastened by ropes 40 feet long to stakes set in the ground, the ground being horizontal. How far are the stakes from the foot of the pole, and what is the inclination of the ropes to the ground? Ans. 34.641 feet; 30° .

3. The top of a lighthouse is 200 feet above the sea level, and the angle of depression to a buoy is $9^\circ 52'8$. Find the horizontal distance of the buoy from the lighthouse. Ans. 1148.3 feet.

4. The horizontal distance from a point to the vertical wall of a tower is 1000 feet, and the angle of elevation of the top is $4^\circ 15'2$. Find the height of the top of the wall above the point. Ans. 74.370 feet.

5. Two points A and B are on the opposite banks of a stream. A line AC at right angles to AB is measured 300 feet long, and the angle ACB is found by measurement to be $62^\circ 30'4$. What is the distance from A to B ? Ans. 576.45 feet.

6. From the top of a lighthouse, 150 feet above the sea level, the angle of depression to a buoy was $12^\circ 10'2$, and that to the shore, measured in the same vertical plane with the buoy, was $62^\circ 14'8$. Find the distance in feet of the buoy from the shore. Ans. Log. Tables, 616.60; Nat. Tables, 616.61.

28. A hill rises 1 foot vertically in a horizontal distance of 30 feet. What is the difference of elevation of two points that are 1000 feet apart, the distance being measured on the ground?

$$\log \tan \alpha = 8.52288 - 10$$

$$\text{Cpl. } T' = 3.53611$$

$$\log \alpha' = 2.05899$$

$$S' = 6.46365 - 10$$

$$\log \sin \alpha = 8.52264 - 10$$

$$\log 1000 = 3.$$

$$\log \text{diff. of elev.} = 1.52264$$

$$\text{diff. of elev.} = 33.315 \text{ feet.}$$

29. The horizontal distance between the two extreme positions of the end of a pendulum 40 inches long is 4 inches. Through what angle does it swing?

$$\text{Half-angle} = 2^\circ 51'.96.$$

$$\text{Ans. } 5^\circ 43'.92.$$

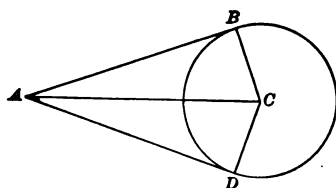


FIG. 28.

30. The angular diameter of the moon is $31'.12$, and its distance is 238 840 miles. Find its diameter in miles.

$$BAD = 31'.12, \text{ and } AC = 238\,840.$$

$$\text{Ans. } 2162.0 \text{ miles.}$$

31. The equatorial horizontal parallax of the sun is $8''.8$, and the radius of the earth is 3963.3 miles. Find the distance of the sun from the earth.

$$BAC = 8''.8, \text{ and } BC = 3963.3.$$

$$\text{Ans. } 92\,896\,000 \text{ miles.}$$

32. A circular chimney 100 feet high is 10 feet in diameter at the base, and 8 feet at the top. Find the angle at the vertex of the cone of which it is a frustum.

$$\text{Half-angle} = 34'.376.$$

$$\text{Ans. } 1^\circ 8'.752.$$

Solve the following triangles, the first two elements being given:

$$33. \ c = 0.02934, \ A = 31^\circ 14'.2. \ \therefore \ B = 58^\circ 45'.8; \ a = 0.015215; \ b = 0.025086.$$

$$34. \ c = 4.6136, \ B = 47^\circ 15'.6. \ \therefore \ A = 42^\circ 44'.4; \ a = 3.1311; \ b = 3.3885.$$

$$35. \ c = 436.53, \ A = 74^\circ 10'.6. \ \therefore \ B = 15^\circ 49'.4; \ a = 419.98; \ b = 119.03.$$

$$36. \ c = 0.96724, \ B = 40^\circ 40'.2. \ \therefore \ A = 49^\circ 19'.8; \ a = 0.73363; \ b = 0.63036.$$

$$37. \ c = 110.97, \ a = 67.291. \ \therefore \ A = 37^\circ 19'.8; \ B = 52^\circ 40'.2; \ b = 88.236.$$

$$38. \ c = 1843.7, \ b = 618.42. \ \therefore \ A = 70^\circ 24'.1; \ B = 19^\circ 35'.9; \ a = 1736.9.$$

$$39. \ c = 8226.5, \ a = 814.33. \ \therefore \ A = 81^\circ 50'.5; \ B = 8^\circ 9'.5; \ b = 116.74.$$

$$40. \ c = 0.03672, \ b = 0.01296. \ \therefore \ A = 69^\circ 19'.9; \ B = 20^\circ 40'.1; \ a = 0.034357.$$

$$41. \ c = 4.8293, \ b = 0.31435. \ \therefore \ A = 86^\circ 16'.1; \ B = 3^\circ 43'.9; \ a = 4.8191.$$

42. $a = 43.148$, $b = 84.107$. $\therefore A = 27^\circ 9'.5$; $B = 62^\circ 50'.5$; $c = 94.530$.
 43. $a = 759.28$, $b = 51.85$. $\therefore A = 86^\circ 5'.6$; $B = 3^\circ 54'.4$; $c = 761.05$.
 44. $a = 7642.5$, $b = 864.7$. $\therefore A = 83^\circ 32'.7$; $B = 6^\circ 27'.3$; $c = 7691.3$.
 45. $a = 0.04326$, $b = 0.54318$. $\therefore A = 4^\circ 33'.2$; $B = 85^\circ 26'.8$; $c = 0.54489$.
 46. $a = 903.64$, $A = 22^\circ 10'.3$. $\therefore B = 67^\circ 49'.7$; $b = 2217.4$; $c = 2394.5$.
 47. $b = 0.47922$, $A = 62^\circ 16'.4$. $\therefore B = 27^\circ 43'.6$; $a = 0.91176$; $c = 1.0300$.
 48. $a = 8.4642$, $B = 30^\circ 16'.4$. $\therefore A = 59^\circ 43'.6$; $b = 4.9409$; $c = 9.80075$.
 49. $b = 18.436$, $B = 65^\circ 15'.6$. $\therefore A = 24^\circ 44'.4$; $a = 8.4954$; $c = 20.299$.

Solve the isosceles triangles (Fig. 16) in the following examples, the first two elements being given :

50. $a = 57.906$, $b = 62.736$. $\therefore \alpha = 57^\circ 12'.05$; $\beta = 65^\circ 35'.9$; $p = 48.673$.
 51. $\alpha = 3.4782$, $\alpha = 20^\circ 20'.6$. $\therefore \beta = 139^\circ 18'.8$; $b = 6.5224$; $p = 1.2091$.
 52. $\alpha = 99.674$, $\beta = 40^\circ 30'.4$. $\therefore \alpha = 69^\circ 44'.8$; $b = 69.008$; $p = 93.510$.
 53. $b = 0.96042$, $\alpha = 70^\circ 10'.4$. $\therefore \beta = 39^\circ 39'.2$; $a = 1.4158$; $p = 1.3319$.
 54. $b = 1146.48$, $\beta = 80^\circ 36'.4$. $\therefore \alpha = 49^\circ 41'.8$; $a = 886.24$; $p = 675.87$.
 55. $a = 87.904$, $p = 46.812$. $\therefore \alpha = 32^\circ 10'.6$; $\beta = 115^\circ 38'.8$; $b = 148.806$.
 56. $b = 6.9044$, $p = 5.7806$. $\therefore \alpha = 59^\circ 9'.2$; $\beta = 61^\circ 41'.6$; $a = 6.7330$.
 57. $p = 18.478$, $\alpha = 37^\circ 19'.8$. $\therefore \beta = 105^\circ 20'.4$; $a = 30.471$; $b = 48.458$.
 58. $p = 0.46424$, $\beta = 100^\circ 36'.8$. $\therefore \alpha = 39^\circ 41'.6$; $a = 0.72690$; $b = 1.11865$.

CHAPTER III.

TRIGONOMETRIC FUNCTIONS OF ANY ANGLE.

30. Generation of Angles. — An angle may be considered as generated by a line revolving about a fixed point, the vertex ;

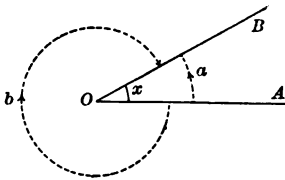


FIG. 24.

thus OA revolving about O in the direction a , to the position OB , describes the angle AOB . The side of the angle *from which* the revolution takes place is called the *initial side*, and that *to which* the describing line moves is called the *terminal side*.

The letters describing the initial side are written first in the symbol of the angle, so that the angle AOB is one in which the motion is from OA to OB .

31. Direction of Measurement. — The revolving line can move from OA to OB either in the direction marked a or in that marked b . The former motion, contrary to that of the hands of a watch, is arbitrarily considered positive and the latter negative. Thus if the angle x , between OA and OB , is 30° , the angle AOB is either $+30^\circ$ or -330° .

Any angle has two measures less than 360° , one positive and the other negative, their numerical sum being 360° .

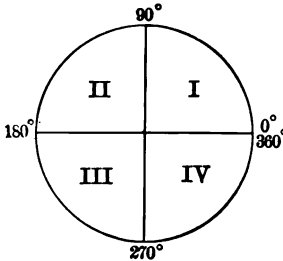


FIG. 25.

32. Quadrants. — For convenience the measuring circle is divided into four parts called *quadrants*, as in the figure. An angle is in the first quadrant when its value lies between 0° and 90° ; in the second, between 90° and 180° ; in the third, between 180°

and 270° ; in the fourth, between 270° and 360° . Angles between 0° and -90° are in the fourth quadrant; between -90° and -180° , in the third; between -180° and -270° , in the second; between -270° and -360° , in the first.

Also, an angle between zero and $\frac{1}{2}\pi$ is in the first quadrant; between $\frac{1}{2}\pi$ and π , in the second; between π and $\frac{3}{2}\pi$, in the third; and between $\frac{3}{2}\pi$ and 2π , in the fourth.

33. Complement and Supplement. — Two angles are said to be complementary when their algebraic sum is 90° , as 60° and 30° , 120° and -30° , 260° and -170° ; and supplementary when their algebraic sum is 180° , as 120° and 60° , 230° and -50° , 300° and -120° .

NOTE. — In Fig. 2, $\frac{a}{h}$ is the sine of B ; that is, it is the sine of the complement of A , and hence it is called the cosine of A .

Since $\frac{1}{2}\pi$ corresponds to 90° , and π to 180° , two angles are complementary when the algebraic sum of their circular measures is $\frac{1}{2}\pi$, and supplementary when it is π .

1. The complement of 200° is $90^\circ - 200^\circ = -110^\circ$.
2. The complement of $90^\circ + x$ is $90^\circ - (90^\circ + x) = -x$.
3. The supplement of 200° is $180^\circ - 200^\circ = -20^\circ$.
4. The supplement of $270^\circ + x$ is $180^\circ - (270^\circ + x) = -90^\circ - x$.
5. The complement of $\frac{9}{16}\pi$ is $\frac{1}{2}\pi - \frac{9}{16}\pi = -\frac{1}{16}\pi$.
6. The supplement of $\frac{5}{8}\pi$ is $\pi - \frac{5}{8}\pi = -\frac{3}{8}\pi$.

Show that the complement of the first angle of each of the following pairs is equal to the second angle:

7. 145° and -55° ; 300° and -210° ; -70° and $+160^\circ$; -200° and $+290^\circ$.
8. $180^\circ - x$ and $-90^\circ + x$; $270^\circ - x$ and $-180^\circ + x$; $360^\circ - x$ and $-270^\circ + x$.
9. $\frac{1}{4}\pi$ and $\frac{1}{4}\pi$; $\frac{3}{8}\pi$ and $-\pi$; $\pi - x$ and $x - \frac{1}{2}\pi$; $\frac{3}{8}\pi + x$ and $-\frac{1}{8}\pi - x$.

Show that the supplement of the first angle of each of the following pairs is equal to the second angle:

10. 145° and 35° ; 225° and -45° ; -160° and 340° ; -70° and 250° .
11. $270^\circ - x$ and $-90^\circ + x$; $90^\circ + x$ and $90^\circ - x$; $x - 90^\circ$ and $270^\circ - x$.
12. $\frac{1}{4}\pi$ and $\frac{3}{4}\pi$; $\frac{5}{8}\pi$ and $-\frac{3}{8}\pi$; $x - \pi$ and $2\pi - x$; $\frac{3}{8}\pi + x$ and $-\frac{1}{8}\pi - x$.

34. General Measure of an Angle. — The line OA may be brought into the position OB by revolving either through the small angle x , or through that angle and then through any

number of complete revolutions in either direction. The

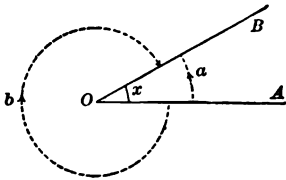


FIG. 26.

general measure of the angle AOB is then not x , but $x + n360^\circ$, where n is any whole number, positive or negative.

The general circular measure of the angle whose circular measure less than 2π is x would be $x + 2n\pi$,

since 2π corresponds to a complete revolution.

1. Show that 1000° is in the fourth quadrant.*

$1000^\circ = 720^\circ + 280^\circ = 2 \times 360^\circ + 280^\circ$, two complete revolutions and 280° beyond; 280° lies in the fourth quadrant.

2. Show that -3000° is in the third quadrant.

$-3000^\circ = -2880^\circ - 120^\circ = 8(-360^\circ) - 120^\circ$, eight complete revolutions and 120° beyond in the negative direction; -120° lies in the third quadrant.

3. Show that $\frac{\pi}{2}(8n + \frac{3}{2})$ is in the first quadrant.

$\frac{\pi}{2}(8n + \frac{3}{2}) = 2n \times 2\pi + \frac{3}{4}\pi$, $2n$ complete revolutions and $\frac{3}{4}\pi$ beyond; $\frac{3}{4}\pi$ is in the first quadrant.

4. Show that 1500° is in the first quadrant, 2690° in the second, 2720° in the third, 2100° in the fourth.

5. Show that -910° is in the second quadrant, -1100° in the fourth, -1400° in the first, -1920° in the third.

6. Show that $\frac{\pi}{5}(10n + 6)$ is in the third quadrant, $\frac{\pi}{3}(12n + 2)$ in the second, $\frac{\pi}{4}(8n + 7)$ in the fourth, $\frac{2}{3}\pi(3n + 2)$ in the third.

7. Show that $\frac{2}{3}\pi(10n - \frac{1}{2})$ is in the fourth quadrant, $\frac{2}{3}\pi(15n - \frac{1}{2})$ in the third, $\frac{2}{3}\pi(-9n - \frac{1}{2})$ in the third, $\frac{2}{3}\pi(10n - 9)$ in the first.

8. Show that $\frac{\pi}{3}(9n + 1)$ will lie in the third or in the first quadrant, according as n is odd or even.

9. Show that the general circular measure of 0° is $2n\pi$, and not $n\pi$.

10. Show that the general circular measure of 90° is $(2n + \frac{1}{2})\pi$; of 180° , $(2n + 1)\pi$; of 270° , $(2n + \frac{3}{2})\pi$.

11. If $x = 60^\circ$, show that one third of the general measure of x will be 20° , 140° , and 260° , the terminal side of the angle for all values of $\frac{1}{3}x$ greater than 260° falling in one of these positions.

We have, using the general measure, $x + n360^\circ$,

$$x = 60^\circ, 420^\circ, 780^\circ, 1140^\circ, 1500^\circ, 1860^\circ, \dots$$

$$\therefore \frac{1}{3}x = 20^\circ, 140^\circ, 260^\circ, 380^\circ, 500^\circ, 620^\circ, \dots$$

$$\text{or} \quad \frac{1}{3}x = 20^\circ, 140^\circ, 260^\circ, \quad 20^\circ, 140^\circ, 260^\circ, \dots$$

if we reduce the values of $\frac{1}{3}x$ that are greater than 360° to others less than 360° by subtracting some multiple of 360° .

* That is, show that when the angle is 1000° the terminal side will lie in the fourth quadrant.

12. If $x = 45^\circ$, show that $\frac{1}{2}x$ will be $15^\circ, 135^\circ, 255^\circ$, three values.
13. If $x = 20^\circ$, show that $\frac{1}{2}x$ will be $5^\circ, 95^\circ, 185^\circ, 275^\circ$, four values.
14. If $x = 60^\circ$, show that $\frac{1}{2}x$ will be $10^\circ, 70^\circ, 130^\circ, 190^\circ, 250^\circ, 310^\circ$, six values.
15. If $x = m^\circ$, show that $\frac{1}{n}x$ will have n values less than 360° , as $\frac{m^\circ}{n}$, $\frac{m^\circ}{n} + \frac{360^\circ}{n}$, $\frac{m^\circ}{n} + \frac{720^\circ}{n}$, ... to $\frac{m^\circ}{n} + \frac{(n-1)360^\circ}{n}$.

35. The definitions of the trigonometric ratios in Art. 8 are applicable only to angles less than 90° . We shall now consider the more general definitions, of which those in Art. 8 are special cases.

36. **Map Drawing by Coördinates.*** — Let $ABCD$ be a field whose map is wanted. From any point O in the field, measure the distances Oa , Ob , Oc , and Od , and also measure the distances aA , bB , cC , and dD , at right angles to $X'OX$. Lay off on the paper a line $X'X$ of indefinite length, and take on it some point O to represent the point O in the field. Lay off Oa according to some convenient scale; thus if Oa were 200 feet, and the scale were 20 feet to 1 inch, we would on the map make Oa 10 inches long. Then draw the line aA perpendicular to OX on the proper side of OX , and lay off on it the distance corresponding to aA according to the same scale, thus locating the point A . The other points would be located in a similar manner.

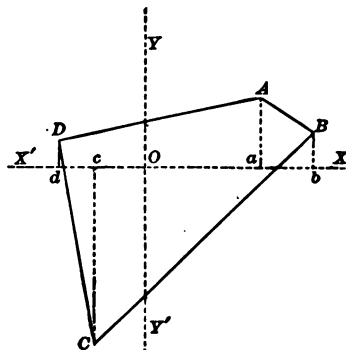


FIG. 27.

Since Oa and Oc are measured from O in contrary directions, and aA and cC are measured on opposite sides of $X'X$, there is danger of laying them off in the wrong direction; hence their directions must be carefully distinguished.

37. **Coördinates.** — The distance Oa , measured along $X'OX$, is called the *abscissa* of the point A ; aA , measured parallel to

* This is called the method of offsets.

$Y'OY$, the *ordinate* of A ; and the two distances Oa and aA , the *coördinates* of A . The line $X'OX$ is called the *axis of abscissas*; the line $Y'OY$, the *axis of ordinates*; and the point O , the *origin of coördinates*.

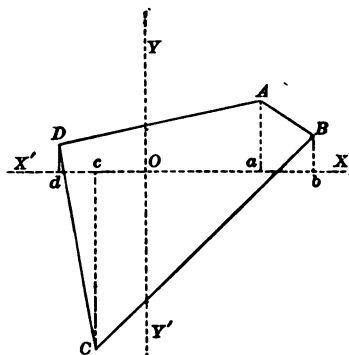


FIG. 28.

The *abscissa* of a point is its distance from the axis of ordinates measured on a line parallel to the axis of abscissas.

The *ordinate* of a point is its distance from the axis of abscissas measured on a line parallel to the axis of ordinates.

The *abscissa* is *positive* when the point is on the *right* of the axis of ordinates, and *negative* when it is on the *left*; the *ordinate*

is *positive* when the point is *above* the axis of abscissas, and *negative* when it is *below*. If we consider the abscissas as measured from $Y'OY$, and the ordinates from $X'OX$, they will be positive when measured to the right and upward respectively.

Using the customary notation for directed lines,* Oc will represent a line measured from O to c , and cO will be measured from c to O . The line cO measured to the right is positive, and Oc to the left is negative. Hence the coördinates of C are Oc and cC , or $-cO$ and $-Cc$. For brevity, the coördinates of a point are written in a parenthesis with a comma between them, the abscissa being written first; thus the point D is called the point (Od, dD) .

The ordinate of any point on $X'OX$ is zero, the abscissa of any point on $Y'OY$ is zero, and both coördinates of the origin are zero.

The signs of the numerical coördinates of points in the different quadrants are as follows :

Quadrant	.	.	.	.	I.	II.	III.	IV.
Abscissa	.	.	.	.	+	-	-	+
Ordinate	.	.	.	.	+	+	-	-

* See Art. 2.

38. Distance of a Point from the Origin.—Represent the abscissa of the point by a , its ordinate by o , and its distance from the origin by h . Then

$$h = \sqrt{a^2 + o^2},$$

since h is the hypotenuse of a right triangle whose sides are a and o . Although h may be either positive or negative, it will be sufficient for our purposes to treat it as being always positive.

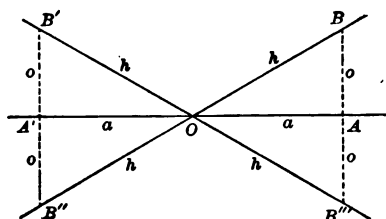


FIG. 29.

39. Trigonometric Ratios.—Take the origin of coördinates at the vertex of the angle and the initial side as the axis of abscissas. From *any* point B on the terminal side of the angle, draw AB perpendicular to the initial side; denote the abscissa OA of the point by a , its ordinate AB by o , and its distance OB from the origin by h . The general definitions of the trigonometric ratios are :

$$\left. \begin{array}{lcl} \text{The } \textit{sine} & \text{of the angle} & = \frac{\text{ordinate}}{\text{distance}} = \frac{o}{h} \\ \text{The } \textit{cosine} & \text{of the angle} & = \frac{\text{abscissa}}{\text{distance}} = \frac{a}{h} \\ \text{The } \textit{tangent} & \text{of the angle} & = \frac{\text{ordinate}}{\text{abscissa}} = \frac{o}{a} \\ \text{The } \textit{cotangent} & \text{of the angle} & = \frac{\text{abscissa}}{\text{ordinate}} = \frac{a}{o} \\ \text{The } \textit{secant} & \text{of the angle} & = \frac{\text{distance}}{\text{abscissa}} = \frac{h}{a} \\ \text{The } \textit{cosecant} & \text{of the angle} & = \frac{\text{distance}}{\text{ordinate}} = \frac{h}{o} \end{array} \right\} \quad (1)$$

NOTE.—The origin is always at the vertex of the angle; the axis of abscissas always coincides with the initial side; and the positive direction of the axis of ordinates is along the line that makes an angle of $+90^\circ$ with the initial side.

Prove that the following equations are true, using Eqs. (1):

$$1. \frac{\sec x}{\sqrt{\sec^2 x - 1}} = \operatorname{cosec} x.$$

$$\sec x = \frac{h}{a};$$

$$\therefore \frac{\sec x}{\sqrt{\sec^2 x - 1}} = \frac{\frac{h}{a}}{\sqrt{\frac{h^2}{a^2} - 1}} = \frac{h}{\sqrt{h^2 - a^2}} = \frac{h}{o} = \operatorname{cosec} x.$$

$$2. \sec x \cos x = 1.$$

$$7. \tan x \cot x = 1.$$

$$3. \operatorname{cosec} x \sin x = 1.$$

$$8. \sin^2 x + \cos^2 x = 1.$$

$$4. \operatorname{cosec}^2 x = 1 + \cot^2 x.$$

$$9. \sec^2 x = 1 + \tan^2 x.$$

$$5. \frac{\tan x}{\sqrt{1 + \tan^2 x}} = \sin x.$$

$$10. \frac{\sqrt{1 + \cot^2 x}}{\cot x} = \sec x.$$

$$6. \frac{\sqrt{\operatorname{cosec}^2 x - 1}}{\operatorname{cosec} x} = \cos x.$$

$$11. \sqrt{\frac{1 + \cot^2 x}{\operatorname{cosec}^2 x - 1}} = \sec x.$$

$$12. (\tan x - \cot x)(\tan x + \cot x) = \frac{h^2(o^2 - a^2)}{o^2 a^2} = \sec^2 x - \operatorname{cosec}^2 x.$$

$$13. (\tan x + \cot x) \sin x \cos x = 1.$$

Construct geometrically the angles, and compute the corresponding ratios in the following examples: *

	Quadrant.	Sin.	Cos.	Tan.	Cot.	Sec.	Cosec.
14. $\sin x = +\frac{3}{5}$.	I.	—	$+\frac{4}{5}$.	$+\frac{3}{4}$.	$+\frac{4}{3}$.	$+\frac{5}{3}$.	$+\frac{5}{4}$.
	II.	—	$-\frac{4}{5}$.	$-\frac{3}{4}$.	$-\frac{4}{3}$.	$-\frac{5}{3}$.	$+\frac{5}{4}$.
15. $\sin x = -\frac{3}{5}$.	III.	—	$-\frac{4}{5}\sqrt{2}$.	$+\frac{1}{2}\sqrt{2}$.	$+2\sqrt{2}$.	$-\frac{1}{2}\sqrt{2}$.	-3.
	IV.	—	$+\frac{4}{5}\sqrt{2}$.	$-\frac{1}{2}\sqrt{2}$.	$-2\sqrt{2}$.	$+\frac{1}{2}\sqrt{2}$.	-3.
16. $\cos x = +\frac{1}{2}$.	I.	$+\frac{1}{2}\sqrt{3}$.	—	$+\sqrt{3}$.	$+\frac{1}{3}\sqrt{3}$.	+2.	$+\frac{2}{3}\sqrt{3}$.
	IV.	$-\frac{1}{2}\sqrt{3}$.	—	$-\sqrt{3}$.	$-\frac{1}{3}\sqrt{3}$.	+2.	$-\frac{2}{3}\sqrt{3}$.
17. $\cos x = -\frac{1}{2}$.	II.	$+\frac{3}{4}\sqrt{2}$.	—	$-2\sqrt{2}$.	$-\frac{1}{2}\sqrt{2}$.	-3.	$+\frac{3}{2}\sqrt{2}$.
	III.	$-\frac{3}{4}\sqrt{2}$.	—	$+2\sqrt{2}$.	$+\frac{1}{2}\sqrt{2}$.	-3.	$-\frac{3}{2}\sqrt{2}$.
18. $\tan x = +\frac{1}{2}$.	I.	$+\frac{1}{3}\sqrt{5}$.	$+\frac{2}{3}\sqrt{5}$.	—	+2.	$+\frac{1}{2}\sqrt{5}$.	$+\sqrt{5}$.
	III.	$-\frac{1}{3}\sqrt{5}$.	$-\frac{2}{3}\sqrt{5}$.	—	+2.	$-\frac{1}{2}\sqrt{5}$.	$-\sqrt{5}$.
19. $\tan x = -2$.	II.	$+\frac{2}{3}\sqrt{5}$.	$-\frac{1}{3}\sqrt{5}$.	—	$-\frac{1}{2}$.	$-\sqrt{5}$.	$+\frac{1}{2}\sqrt{5}$.
	IV.	$-\frac{2}{3}\sqrt{5}$.	$+\frac{1}{3}\sqrt{5}$.	—	$-\frac{1}{2}$.	$+\sqrt{5}$.	$-\frac{1}{2}\sqrt{5}$.

* See Arts. 11 and 15. If $\sin x$ is positive, o must be positive, since h is always positive, and the angle lies in quadrants I. and II.

		Quadrant.	Sin.	Cos.	Tan.	Cot.	Sec.	Cosec.
20.	$\cot x = +\frac{4}{3}$.	I.	$+\frac{3}{5}$.	$+\frac{4}{5}$.	$+\frac{4}{3}$.	—	$+\frac{5}{4}$.	$+\frac{5}{3}$.
		III.	$-\frac{3}{5}$.	$-\frac{4}{5}$.	$+\frac{4}{3}$.	—	$-\frac{5}{4}$.	$-\frac{5}{3}$.
21.	$\cot x = -3$.	II.	$+\frac{1}{10}\sqrt{10}$.	$-\frac{3}{10}\sqrt{10}$.	$-\frac{1}{3}$.	—	$-\frac{1}{3}\sqrt{10}$.	$+\sqrt{10}$.
		IV.	$-\frac{1}{10}\sqrt{10}$.	$+\frac{3}{10}\sqrt{10}$.	$-\frac{1}{3}$.	—	$+\frac{1}{3}\sqrt{10}$.	$-\sqrt{10}$.
22.	$\sec x = +3$.	I.	$+\frac{3}{5}\sqrt{2}$.	$+\frac{1}{5}$.	$+2\sqrt{2}$.	$+\frac{1}{2}\sqrt{2}$.	—	$+\frac{3}{2}\sqrt{2}$.
		IV.	$-\frac{3}{5}\sqrt{2}$.	$+\frac{1}{5}$.	$-2\sqrt{2}$.	$-\frac{1}{2}\sqrt{2}$.	—	$-\frac{3}{2}\sqrt{2}$.
23.	$\sec x = -\frac{5}{3}$.	II.	$+\frac{4}{5}$.	$-\frac{3}{5}$.	$-\frac{4}{3}$.	$-\frac{3}{4}$.	—	$+\frac{5}{4}$.
		III.	$-\frac{4}{5}$.	$-\frac{3}{5}$.	$+\frac{4}{3}$.	$+\frac{3}{4}$.	—	$-\frac{5}{4}$.
24.	$\operatorname{cosec} x = +\frac{1}{3}$.	I.	$+\frac{5}{13}$.	$+\frac{12}{13}$.	$+\frac{5}{12}$.	$+\frac{13}{5}$.	$+\frac{13}{5}$.	—
		II.	$+\frac{5}{13}$.	$-\frac{12}{13}$.	$-\frac{5}{12}$.	$-\frac{13}{5}$.	$-\frac{13}{5}$.	—
25.	$\operatorname{cosec} x = -\frac{2}{7}$.	III.	$-\frac{7}{25}$.	$-\frac{24}{25}$.	$+\frac{7}{24}$.	$+\frac{24}{7}$.	$-\frac{25}{24}$.	—
		IV.	$-\frac{7}{25}$.	$+\frac{24}{25}$.	$-\frac{7}{24}$.	$-\frac{24}{7}$.	$+\frac{25}{24}$.	—

40. Trigonometric Functions. — One quantity is said to be a function of another when it depends upon the latter for its value. Thus, if $y = \sin x$, y is a function of x , since it depends upon x for its value, any change in the value of x producing a change in the value of y .

The *trigonometric functions* are the sine, cosine, tangent, cotangent, secant, cosecant, versed sine, covered sine, and suversed sine. The last three are defined by the equations :

$$\left. \begin{array}{ll} \text{The versed sine is} & \text{vers } x = 1 - \cos x. \\ \text{The covered sine is} & \text{covers } x = 1 - \sin x. \\ \text{The suversed sine is} & \text{suvers } x = 1 + \cos x. \end{array} \right\} \quad (1)$$

41. Geometrical Representation of the Functions. — In Fig. 30 let the radius OB , of the circle described about the vertex O of the angle AOB as a center, be unity, and let the angle AOY be equal to 90° . NM and FD are tangent to the circle at X and Y respectively; the triangles OAB , OXM , and OYD , are right-angled; and the angle YDO is equal to the given angle AOB . Then the trigonometric functions of the angle AOB are represented by the lines shown in the figure. For, in Figs. 2 and 29, B is any point on the terminal side OB of the angle AOB , and therefore we may choose the position of B so that OB , or h , shall be equal to unity. Comparing Fig. 30 with Figs. 2 and 29, and using the definitions in Arts. 8, 39, and 40, we see that

$$\sin AOB = \frac{o}{h} = \frac{AB}{OB} = AB.$$

$$\cos AOB = \frac{a}{h} = \frac{OA}{OB} = OA.$$

$$\tan AOB = \frac{o}{a} = \frac{AB}{OA} = \frac{XM}{OX} = XM.$$

$$\cot AOB = \frac{a}{o} = \frac{OA}{AB} = \frac{CB}{OC} = \frac{YD}{OY} = YD.$$

$$\sec AOB = \frac{h}{a} = \frac{OB}{OA} = \frac{OM}{OX} = OM.$$

$$\operatorname{cosec} AOB = \frac{h}{o} = \frac{OB}{AB} = \frac{OB}{OC} = \frac{OD}{OY} = OD.$$

$$\operatorname{vers} AOB = 1 - \cos AOB = OX - OA = AX.$$

$$\operatorname{covers} AOB = 1 - \sin AOB = OY - OC = CY.$$

$$\operatorname{suvers} AOB = 1 + \cos AOB = X'O + OA = X'A.$$

The trigonometric functions are *ratios*,—pure numbers,—and are represented by these lines in the circle whose radius is unity; that is, they are actually equal to the ratios of these lines to the radius.

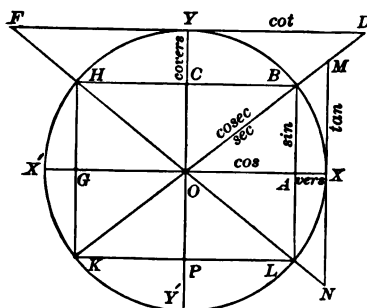


FIG. 80.

If, with a radius of unity and the vertex of the angle as the center, a circle be described and two tangents be drawn, one where the initial side OA cuts the circle, and the other at a distance of $+90^\circ$ from this point (at X and Y respectively), the trigonometric functions will be represented as follows:

The sine of an angle will be the perpendicular distance from the point where the terminal side of the angle cuts the circle, to the initial side, produced if necessary; positive when it is above, and negative when below, the initial side. Thus $\sin AOB = AB$, $\sin AOH = GH$, $\sin AOK = GK$, $\sin AOL = AL$. AB and GH , above $X'OX$, are positive, while GK and AL are negative, being below $X'OX$. The sine is therefore positive when

the angle is in the first or second quadrant, and negative when it is in the third or fourth.

The cosine will be the distance from the center to the foot of the sine; positive when measured to the right, and negative to the left, of the center. Thus $\cos AOB = OA$, $\cos AOH = OG$, $\cos AOK = OG$, $\cos AOL = OA$. OA , measured to the right of the center, is positive, while OG , measured to the left, is negative. The cosine is therefore positive when the angle is in the first or fourth quadrant, and negative when it is in the second or third.*

The tangent will be the distance along the line tangent to the circle at the point where the initial side cuts the circle, from this point to the point where this tangent is cut by the terminal side of the angle, produced if necessary; positive when measured above, and negative when below, the initial side. Thus $\tan AOB = XM$, $\tan AOH = XN$, $\tan AOK = XM$, $\tan AOL = XN$. XM , above $X'OX$, is positive, and XN , below $X'OX$, is negative. Therefore the tangent is positive when the angle is in the first or third quadrant, and negative when it is in the second or fourth.

The cotangent will be the distance along the second tangent (FYD) from the point of tangency to the point where this line is cut by the terminal side of the angle, produced if necessary; positive when measured to the right, and negative to the left, of the point of tangency. Thus $\cot AOB = YD$, $\cot AOH = YF$, $\cot AOK = YD$, $\cot AOL = YF$. YD , measured to the right, is positive, and YF , measured to the left, is negative. Therefore the cotangent is positive when the angle is in the first or third quadrant, and negative when it is in the second or fourth.

NOTE. — The positive directions of measurement are above $X'OX$ and to the right of $Y'OY$, and the negative are below $X'OX$ and to the left of $Y'OY$.

The secant will be the distance from the center along the terminal side of the angle, produced if necessary, to its point of intersection with the tangent at the point of intersection of the initial side with the circle; positive when measured along the side itself, and negative when along the side produced. Thus $\sec AOB = OM$, $\sec AOH = ON$, $\sec AOK = OM$, $\sec AOL = ON$.

* The foot of the sine is the point where the perpendicular line representing the sine cuts the initial side, produced if necessary.

Since $\sec AOB$ and $\sec AOL$ are measured along the terminal side itself, they are positive. The terminal sides (OH and OK) of the angles AOH and AOK must be produced in order that they may intersect the tangent line NM , and therefore $\sec AOH$ and $\sec AOK$ are negative. Hence the secant is positive when the angle is in the first or fourth quadrant, and negative when it is in the second or third.

The cosecant will be the distance from the center along the terminal side, produced if necessary, to its intersection with the second tangent, FYD; positive when measured along the side itself, and negative when along the side produced. Thus $\operatorname{cosec} AOB = OD$, $\operatorname{cosec} AOH = OF$, $\operatorname{cosec} AOK = OD$, $\operatorname{cosec} AOL = OF$. Since $\operatorname{cosec} AOB$ and $\operatorname{cosec} AOH$ are measured along the terminal side itself, they are positive, while $\operatorname{cosec} AOK$ and $\operatorname{cosec} AOL$, measured along the side produced, are negative. Therefore the cosecant is positive when the angle is in the first or second quadrant, and negative when it is in the third or fourth.

The versed sine ($1 - \cos x$) will be the distance from the foot of the sine to the point where the initial side cuts the circle; always positive, because $\cos x$ can never be greater than the radius, or unity. Thus $\operatorname{vers} AOB = AX$, $\operatorname{vers} AOH = GX$, $\operatorname{vers} AOK = GX$, $\operatorname{vers} AOL = AX$.

The coversed sine ($1 - \sin x$) will be the distance from the point C or P, where a line drawn through the point of intersection of the terminal side and the circle parallel to the initial side cuts $Y'OY$, to the point Y; always positive, since $\sin x$ can never be greater than the radius, or unity. Thus $\operatorname{covers} AOB = CY$, $\operatorname{covers} AOH = CY$, $\operatorname{covers} AOK = PY$, $\operatorname{covers} AOL = PY$.

The suversed sine ($1 + \cos x$) will be the distance from the point X' , where the initial side produced cuts the circle, to the foot of the sine; always positive, since $\cos x$ can never be algebraically less than minus unity. Thus $\operatorname{suvers} AOB = X'A$, $\operatorname{suvers} AOH = X'G$, $\operatorname{suvers} AOK = X'G$, $\operatorname{suvers} AOL = X'A$.

NOTE. — These lines represent the trigonometric functions, only when the radius of the circle is unity. If the radius differs from unity, the functions are equal to the lengths of these lines divided by the radius.

42. Changes in the Values of the Functions. — Let OX be the initial side of the angle, and let the terminal side first

coincide with OX , and then, in revolving about O , come into the positions OM , OY , OH , OX' , OK , OY' , ON , and OX , and let us consider the resulting changes in the values of the sine and of the tangent.

The sine of 0° , the terminal side coinciding with OX , is zero. As the angle increases, the sine, being positive, also increases ($\sin AOB = AB$), until at 90° it is equal to the radius, or $+1$ ($\sin AOY = OY$). The sine then decreases ($\sin AOH = GH$), still being positive; and at 180° it is zero, the terminal side coinciding with OX' . The sine then becomes negative, and decreases algebraically, increasing numerically ($\sin AOK = GK$), until at 270° it is equal to the radius, or -1 ($\sin AOY' = OY'$). It then increases algebraically, decreasing numerically ($\sin AOL = AL$); and at 360° it again becomes zero.

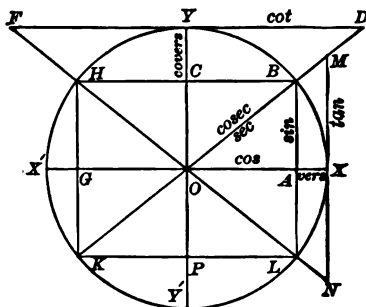


FIG. 81.

The tangent of 0° is zero; the tangent then becomes positive, and at 90° it is infinite, the terminal side being parallel to XM ; then negative, and at 180° it is zero; then positive, and at 270° it is infinite; then negative, and at 360° it is zero. Just before the terminal side reaches the position OY , the tangent is positive, and just after, it is negative; therefore the tangent of 90° is $\pm\infty$, the upper sign being that of the function of an angle a little less than 90° .

The table gives the values of the functions of 0° , 90° , 180° , 270° , and 360° , and their signs in quadrants I., II., III., and IV.:

	0° .	I.	90° .	II.	180° .	III.	270° .	IV.	360° .
sin.	0	+	+1	+	0	-	-1	-	0
cos.	+1	+	0	-	-1	-	0	+	+1
tan.	0	+	∞	-	0	+	∞	-	0
cot.	∞	+	0	-	∞	+	0	-	∞
sec.	+1	+	∞	-	-1	-	∞	+	+1
cosec.	∞	+	+1	+	∞	-	-1	-	∞

43. Limiting Values of the Functions. — The sine and cosine may have any value between $+1$ and -1 , but they cannot have a value numerically greater than unity.

The tangent and cotangent may have any value between $+\infty$ and $-\infty$; that is, no matter what a number may be, there will always be some angle that will have that number as the value of its tangent, and another having it as its cotangent.

The secant and cosecant may have any value between $+1$ and $+\infty$, or -1 and $-\infty$; but they cannot have a value numerically less than unity.

The versed sine, covered sine, and suversed sine may have any value between zero and $+2$.

NOTE. — In the first quadrant, all the functions are positive, and the sine, tangent, and secant increase as the angle increases; while the cosine, cotangent, and cosecant decrease as the angle increases.

NOTE. — The functions change signs only when they pass through the values zero and infinity.

44. Graphical Representation of the Functions. — Let the distance OL represent 360° , so that 1° is represented by $\frac{1}{360} OL$. At C , such that $OC = \frac{1}{2} OL$, draw a line perpendicular to OL , and

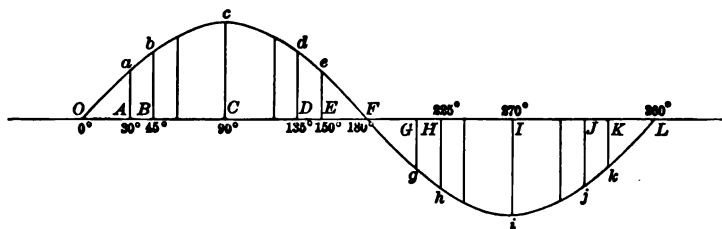


FIG. 82.

lay off on it any convenient distance Cc , to represent the sine of 90° , above the line OL , since $\sin 90^\circ = +1$. At A , such that $OA = \frac{1}{12} OL$, lay off $Aa = \frac{1}{2} Cc$, since $\sin 30^\circ = +\frac{1}{2}$; at B , such that $OB = \frac{1}{8} OL$, lay off $Bb = Cc\sqrt{\frac{1}{2}}$, since $\sin 45^\circ = +\sqrt{\frac{1}{2}}$; at H , such that $OH = \frac{5}{8} OL$, lay off $Hh = Cc\sqrt{\frac{1}{2}}$, below OL , since $\sin 225^\circ = -\sqrt{\frac{1}{2}}$; and so on, locating as many points a, b, c, h , etc., as may be necessary. Draw a smooth curve through $O, a, b, c, d, e, F, h, i, j, k, L$, and we have the *sinusoid*, in which the

abscissas correspond to the angles, and the ordinates to their sines.

We might have taken OL equal to the circumference of the circle whose radius is unity, and Cc equal to this radius. The scale would then have been the same for both the ordinates and the abscissas.

The graphical representations of the other functions may be constructed in a similar manner.

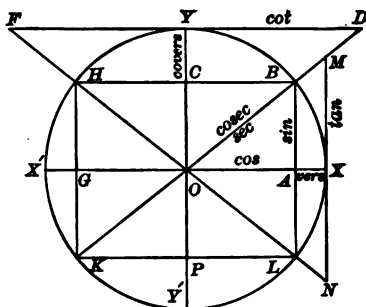


FIG. 33.

45. Two Angles correspond to Any Given Function. — In Fig. 33 let the arcs YB , YH , $Y'K$, and $Y'L$ be equal; therefore the arcs XB , $X'H$, $X'K$, and XL are equal. Hence

$$AB = GH = OC; AL = GK = OP; OM = ON; OD = OF.$$

OC is not equal to OP since they have contrary signs, OC being positive and OP negative on account of their directions.

$$AB = \sin XOB; GH = \sin XOH;$$

$$\therefore \sin XOB = \sin XOH.$$

$$GK = \sin XOK; AL = \sin XOL;$$

$$\therefore \sin XOK = \sin XOL.$$

Therefore two angles that differ by equal amounts from 90° , or from 270° , will have the same sine; thus $\sin(90^\circ + 2^\circ) = \sin(90^\circ - 2^\circ)$, and $\sin(270^\circ + 3^\circ) = \sin(270^\circ - 3^\circ)$.

NOTE. — The two angles corresponding to a given function may be *identical*; thus, if $\sin x = +1$, the only value of x is 90° , or $90^\circ - 0^\circ$ and $90^\circ + 0^\circ$.

$$\text{Again } OA = \cos XOB = \cos XOL;$$

$$\text{and } OG = \cos XOH = \cos XOK.$$

Therefore two angles differing by equal amounts from 0° , from 180° , or from 360° , will have the same cosine; thus $\cos(-5^\circ) = \cos 5^\circ$, $\cos(180^\circ + 5^\circ) = \cos(180^\circ - 5^\circ)$, and $\cos(360^\circ - 10^\circ) = \cos 10^\circ$.

$$\text{Also } XM = \tan XOB = \tan XOK;$$

$$\text{and } XN = \tan XOH = \tan XOL.$$

Therefore two angles differing from each other by 180° will have the same tangent; thus $\tan 140^\circ = \tan 320^\circ$.

Again $YD = \cot XOB = \cot XOK$;
and $YF = \cot XOH = \cot XOL$.

Therefore two angles differing from each other by 180° will have the same cotangent; thus $\cot 200^\circ = \cot 20^\circ$.

Also $+OM = \sec XOB$; $+ON = \sec XOL$;
 $\therefore \sec XOB = \sec XOL$.
 $-OM = \sec XOK$; $-ON = \sec XOH$;
 $\therefore \sec XOK = \sec XOH$.

Therefore two angles differing by equal amounts from 0° , from 180° , or from 360° , will have the same secant; thus $\sec(-5^\circ) = \sec 5^\circ$, $\sec(180^\circ - 3^\circ) = \sec(180^\circ + 3^\circ)$, and $\sec(360^\circ - 5^\circ) = \sec 5^\circ$.

Again $+OD = \operatorname{cosec} XOB$; $+OF = \operatorname{cosec} XOH$;
 $\therefore \operatorname{cosec} XOB = \operatorname{cosec} XOH$.
 $-OD = \operatorname{cosec} XOK$; $-OF = \operatorname{cosec} XOL$;
 $\therefore \operatorname{cosec} XOK = \operatorname{cosec} XOL$.

Therefore two angles differing by equal amounts from 90° , or from 270° , will have the same cosecant; thus $\operatorname{cosec}(90^\circ + 10^\circ) = \operatorname{cosec}(90^\circ - 10^\circ)$, and $\operatorname{cosec}(270^\circ - 60^\circ) = \operatorname{cosec}(270^\circ + 60^\circ)$.

The four angles XOB , XOH , XOK , and XOL , have the same functions numerically. Thus if $\sin x = \pm \frac{1}{2}$, x will be 30° , 150° , 210° , and 330° ; the first two corresponding to the value $+\frac{1}{2}$, and the last two to $-\frac{1}{2}$.

EXAMPLES.

- | | |
|---|---------------------------|
| 1. What angle has the same sine as 140° ? | <i>Ans.</i> 40° . |
| 2. What angle has the same sine as 220° ? | <i>Ans.</i> 320° . |
| 3. What angle has the same cosine as 330° ? | <i>Ans.</i> 30° . |
| 4. What angle has the same cosine as 220° ? | <i>Ans.</i> 140° . |
| 5. What angle has the same tangent as 230° ? | <i>Ans.</i> 50° . |
| 6. What angle has the same tangent as 300° ? | <i>Ans.</i> 120° . |
| 7. What angle has the same cotangent as 240° ? | <i>Ans.</i> 60° . |
| 8. What angle has the same cotangent as 110° ? | <i>Ans.</i> 290° . |
| 9. What angle has the same secant as 315° ? | <i>Ans.</i> 45° . |
| 10. What angle has the same secant as 160° ? | <i>Ans.</i> 200° . |
| 11. What angle has the same cosecant as 110° ? | <i>Ans.</i> 70° . |
| 12. What angle has the same cosecant as 300° ? | <i>Ans.</i> 240° . |

Find the values of θ less than 360° in Exs. (13-24) : *

13. $\sin \theta = -\sin 200^\circ$. *Ans.* 20° . 19. $\cot \theta = -\cot 105^\circ$. *Ans.* 75° .
 14. $\sin \theta = -\sin 100^\circ$. *Ans.* 260° . 20. $\cot \theta = -\cot 205^\circ$. *Ans.* 155° .
 15. $\cos \theta = -\cos 150^\circ$. *Ans.* 30° . 21. $\sec \theta = -\sec 140^\circ$. *Ans.* 40° .
 16. $\cos \theta = -\cos 300^\circ$. *Ans.* 120° . 22. $\sec \theta = -\sec 325^\circ$. *Ans.* 145° .
 17. $\tan \theta = -\tan 350^\circ$. *Ans.* 10° . 23. $\operatorname{cosec} \theta = -\operatorname{cosec} 120^\circ$. *Ans.* 240° .
 18. $\tan \theta = -\tan 230^\circ$. *Ans.* 130° . 24. $\operatorname{cosec} \theta = -\operatorname{cosec} 355^\circ$. *Ans.* 5° .

25. $\cos 3\theta = +\frac{1}{2}\sqrt{3}$. Find three values of θ less than 180° .

3θ may be 30° , or 330° , or these values plus any number of circumferences;

$$\therefore 3\theta = 30^\circ, 390^\circ, 750^\circ, \dots, 330^\circ, 690^\circ, 1050^\circ, \dots$$

$$\therefore \theta = 10^\circ, 130^\circ, 250^\circ, \dots, 110^\circ, 230^\circ, 350^\circ, \dots$$

$$\text{Ans. } \theta = 10^\circ, 110^\circ, 130^\circ.$$

26. $\sin 2\theta = -\frac{1}{2}$. Find four values of θ less than 360° .

$$\text{Ans. } 105^\circ, 285^\circ, 165^\circ, 345^\circ.$$

27. $\tan 3\theta = -1$. Find six values of θ less than 360° .

$$\text{Ans. } 45^\circ, 165^\circ, 285^\circ, 105^\circ, 225^\circ, 345^\circ.$$

28. $\sec 5\theta = -2$. Find five values of θ less than 180° .

$$\text{Ans. } 24^\circ, 96^\circ, 168^\circ, 48^\circ, 120^\circ.$$

29. $\cot 5\theta = +1$. Find five values of θ less than 180° .

$$\text{Ans. } 9^\circ, 81^\circ, 153^\circ, 45^\circ, 117^\circ.$$

30. $\cos 4\theta = -\frac{1}{2}$. Find four values of θ less than 180° .

$$\text{Ans. } 30^\circ, 120^\circ, 60^\circ, 150^\circ.$$

31. $\sin \theta = \frac{1}{2}$. Show that the general measure of θ is $(2n + \frac{1}{2})\pi \pm \frac{1}{2}\pi$.
 $\theta = 30^\circ$ and 150° , or $90^\circ - 60^\circ$ and $90^\circ + 60^\circ$, or $90^\circ \pm 60^\circ$, or $\frac{1}{2}\pi \pm \frac{1}{2}\pi$.

But the general measures of θ are these values increased by any number (n) of circumferences. $\therefore \theta = 2n\pi + \frac{1}{2}\pi \pm \frac{1}{2}\pi = (2n + \frac{1}{2})\pi \pm \frac{1}{2}\pi$.

32. $\sin \theta = +\frac{1}{2}\sqrt{2}$, $\tan \theta = -1$; the general measure of θ is $2n\pi + \frac{1}{2}\pi$.

Note that θ is in the second quadrant, since its sine is positive and its tangent is negative.

33. $\cos \theta = -\frac{1}{2}$, $\operatorname{cosec} \theta = +\frac{3}{2\sqrt{2}}$; the general measure of θ is $2n\pi + \theta'$,

where θ' is the value of θ that lies between $\frac{1}{2}\pi$ and π .

34. $\cos \theta = -\frac{1}{2}$; the general measure of θ is $(2n + 1)\pi \pm \frac{1}{2}\pi$.

35. $\sin 2\theta = +\frac{1}{2}$; the general measures of θ are $(2n + \frac{1}{4})\pi \pm \frac{1}{4}\pi$, and $(2n + \frac{3}{4})\pi \pm \frac{1}{4}\pi$.

36. $\cos 3\theta = -\frac{1}{2}$; the general measures of θ are $(2n + \frac{1}{3})\pi \pm \frac{1}{3}\pi$, $(2n + 1)\pi \pm \frac{1}{3}\pi$, and $(2n + \frac{2}{3})\pi \pm \frac{1}{3}\pi$.

Construct geometrically (Art. 11) the two angles when

37. $\sin x = +\frac{1}{2}$. 41. $\tan x = +2$. 45. $\sec x = +3$.
 38. $\sin x = -\frac{1}{2}$. 42. $\tan x = -\frac{1}{2}$. 46. $\sec x = -\frac{1}{2}$.
 39. $\cos x = +\frac{2}{3}$. 43. $\cot x = +\frac{2}{3}$. 47. $\operatorname{cosec} x = +6$.
 40. $\cos x = -\frac{2}{3}$. 44. $\cot x = -\frac{2}{3}$. 48. $\operatorname{cosec} x = -\frac{2}{3}$.

* Only one of the two answers is given.

CHAPTER IV.

RELATIONS BETWEEN THE FUNCTIONS OF ONE ANGLE.

46. Relations between the Functions of One Angle.

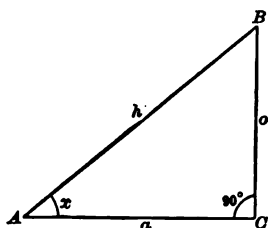


FIG. 34.

$$o^2 + a^2 = h^2; \quad \therefore \frac{o^2}{h^2} + \frac{a^2}{h^2} = 1;$$

$$\therefore \sin^2 x + \cos^2 x = 1. \quad (1)$$

$$\tan x = \frac{o}{a} = \frac{\frac{o}{h}}{\frac{a}{h}} = \frac{\sin x}{\cos x};$$

$$\therefore \tan x = \frac{\sin x}{\cos x}; \quad (2)$$

$$\cot x = \frac{a}{o} = \frac{1}{\tan x}; \quad (3)$$

or, $\cot x = \frac{\cos x}{\sin x}. \quad (4)$

$$h^2 = a^2 + o^2; \quad \therefore \frac{h^2}{a^2} = 1 + \frac{o^2}{a^2};$$

$$\therefore \sec^2 x = 1 + \tan^2 x. \quad (5)$$

$$h^2 = o^2 + a^2; \quad \therefore \frac{h^2}{o^2} = 1 + \frac{a^2}{o^2};$$

$$\therefore \operatorname{cosec}^2 x = 1 + \cot^2 x. \quad (6)$$

$$\sec x = \frac{h}{a} = \frac{1}{\cos x}. \quad (7)$$

$$\sec x = \frac{h}{o} = \frac{1}{\sin x}. \quad (8)$$

$$\operatorname{vers} x = 1 - \cos x. \quad (9)$$

$$\operatorname{covers} x = 1 - \sin x. \quad (10)$$

$$\operatorname{suvers} x = 1 + \cos x. \quad (11)$$

NOTE. — These formulas may be easily remembered by the use of Fig. 30, where

$$AB^2 + OA^2 = OB^2, \text{ or } \sin^2 x + \cos^2 x = 1.$$

$$\tan x = \frac{XM}{OX} = \frac{AB}{OA}, \text{ or } \tan x = \frac{\sin x}{\cos x}.$$

$$\cot x = \frac{YD}{OY} = \frac{CB}{OC}, \text{ or } \cot x = \frac{\cos x}{\sin x}.$$

$$OM^2 = OX^2 + XM^2, \text{ or } \sec^2 x = 1 + \tan^2 x.$$

$$OD^2 = OY^2 + YD^2, \text{ or } \operatorname{cosec}^2 x = 1 + \cot^2 x.$$

47. To express One Function in Terms of Each of the Others.—Suppose that we wish to find expressions for $\sin x$ that shall contain only $\cos x$, $\tan x$, $\cot x$, $\sec x$, and $\operatorname{cosec} x$ respectively. From the preceding article we have:

$$\sin^2 x + \cos^2 x = 1, \text{ and } \operatorname{cosec} x = \frac{1}{\sin x};$$

$$\therefore \sin x = \pm \sqrt{1 - \cos^2 x}$$

and

$$\sin x = \frac{1}{\operatorname{cosec} x}.$$

The other expressions are derived from these as follows:

$$\sin x = \frac{1}{\operatorname{cosec} x} = \pm \frac{1}{\sqrt{1 + \cot^2 x}}, \text{ from (6).}$$

$$\therefore \sin x = \pm \frac{1}{\sqrt{1 + \cot^2 x}} = \pm \frac{1}{\sqrt{1 + \frac{1}{\tan^2 x}}} = \pm \frac{\tan x}{\sqrt{1 + \tan^2 x}}, \text{ from (3).}$$

$$\therefore \sin x = \pm \frac{\tan x}{\sqrt{1 + \tan^2 x}} = \pm \frac{\sqrt{\sec^2 x - 1}}{\sec x}, \text{ from (5).}$$

The double signs are due to the fact that there are two angles corresponding to any given function; thus if $\cos x = \frac{1}{2}$, the angle might be either in the first or in the fourth quadrant, and the sine would be positive in the first case and negative in the second. It will be seen that if any one of the functions is given, all the others found from it will have the double sign, except its reciprocal.

In the same way it may be shown that*

$$\begin{aligned}\cos x &= \frac{\sqrt{1 - \sin^2 x}}{\sqrt{1 + \tan^2 x}} = \frac{1}{\sqrt{1 + \tan^2 x}} = \frac{\cot x}{\sqrt{1 + \cot^2 x}} = \frac{1}{\sec x} = \frac{\sqrt{\operatorname{cosec}^2 x - 1}}{\operatorname{cosec} x} \\ \tan x &= \frac{\sin x}{\sqrt{1 - \sin^2 x}} = \frac{\sqrt{1 - \cos^2 x}}{\cos x} = \frac{1}{\cot x} = \frac{\sqrt{\sec^2 x - 1}}{\sec x} = \frac{1}{\sqrt{\operatorname{cosec}^2 x - 1}} \\ \cot x &= \frac{\sqrt{1 - \sin^2 x}}{\sin x} = \frac{\cos x}{\sqrt{1 - \cos^2 x}} = \frac{1}{\tan x} = \frac{1}{\sqrt{\sec^2 x - 1}} = \sqrt{\operatorname{cosec}^2 x - 1} \\ \sec x &= \frac{1}{\sqrt{1 - \sin^2 x}} = \frac{1}{\cos x} = \sqrt{1 + \tan^2 x} = \frac{\sqrt{1 + \cot^2 x}}{\cot x} = \frac{\operatorname{cosec} x}{\sqrt{\operatorname{cosec}^2 x - 1}} \\ \operatorname{cosec} x &= \frac{1}{\sin x} = \frac{1}{\sqrt{1 - \cos^2 x}} = \frac{\sqrt{1 + \tan^2 x}}{\tan x} = \sqrt{1 + \cot^2 x} = \frac{\sec x}{\sqrt{\sec^2 x - 1}}\end{aligned}$$

If any one of the functions is given, the others may be found from these formulas. It is easier in general to find first the sine and cosine, and then to find the others.

48. Find the Unknown Functions in the Following :

1. $\tan x = -\frac{3}{4}$, x being in the fourth quadrant. Compute the numerical values of the ratios by the method of Art. 15, and then select the proper signs for the functions in the fourth quadrant. Thus let

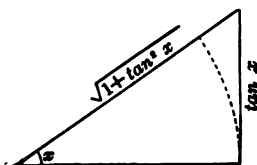


FIG. 35.

$$o = 3, \quad a = 4, \quad h = 5;$$

$$\therefore \sin x = -\frac{3}{5}, \quad \cos x = +\frac{4}{5},$$

$$\cot x = -\frac{4}{3}, \quad \sec x = +\frac{5}{4}, \quad \operatorname{cosec} x = -\frac{5}{3}.$$

2. $\tan x = 2$, x being in the third quadrant. Then

$$\sin x = \pm \frac{\tan x}{\sqrt{1 + \tan^2 x}} = -\frac{2}{\sqrt{5}}, \quad \cos x = \pm \frac{1}{\sqrt{1 + \tan^2 x}} = -\frac{1}{\sqrt{5}}.$$

These convenient formulas may be easily remembered from Fig. 35. Knowing $\sin x$ and $\cos x$, we have

$$\cot x = \frac{1}{\tan x} = +\frac{1}{2}; \quad \sec x = \frac{1}{\cos x} = -\sqrt{5};$$

$$\operatorname{cosec} x = \frac{1}{\sin x} = -\frac{1}{2}\sqrt{5}.$$

* The radicals should be taken with the double sign.

3. $\cot x = -2$, x being in the second quadrant.

$$\therefore \operatorname{cosec} x = \pm \sqrt{1 + \cot^2 x} = +\sqrt{5}; \quad \sin x = \frac{1}{\operatorname{cosec} x} = +\frac{1}{\sqrt{5}};$$

$$\cos x = \pm \sqrt{1 - \sin^2 x} = -\frac{2}{\sqrt{5}}; \quad \tan x = \frac{1}{\cot x} = -\frac{1}{2};$$

$$\sec x = \frac{1}{\cos x} = -\frac{1}{2}\sqrt{5}.$$

4. $\sec x = -\frac{17}{8}$, x being in the third quadrant.

$$\therefore \cos x = \frac{1}{\sec x} = -\frac{8}{17}; \quad \sin x = \pm \sqrt{1 - \cos^2 x} = -\frac{15}{17};$$

$$\tan x = \frac{\sin x}{\cos x} = +\frac{15}{8}; \quad \cot x = +\frac{8}{15}; \quad \operatorname{cosec} x = -\frac{17}{15}.$$

5. $\sin x = -\frac{4}{5}$, x being in the third quadrant.

$$\therefore \cos x = -\frac{3}{5}; \quad \tan x = +\frac{4}{3}; \quad \cot x = +\frac{3}{4}; \quad \sec x = -\frac{5}{3}; \quad \operatorname{cosec} x = -\frac{5}{4}.$$

6. $\cos x = +\frac{4}{5}$, x being in the fourth quadrant.

$$\therefore \sin x = -\frac{3}{5}\sqrt{5}; \quad \tan x = -\frac{3}{4}\sqrt{5}; \quad \cot x = -\frac{4}{3}\sqrt{5}; \quad \sec x = +\frac{5}{4};$$

$$\operatorname{cosec} x = -\frac{5}{3}\sqrt{5}.$$

7. $\tan x = -\frac{5}{12}$, x being in the second quadrant.

$$\therefore \sin x = +\frac{5}{13}; \quad \cos x = -\frac{12}{13}; \quad \cot x = -\frac{12}{5}; \quad \sec x = -\frac{13}{12}; \quad \operatorname{cosec} x = +\frac{13}{5}.$$

8. $\cot x = +\frac{7}{24}$, x being in the third quadrant.

$$\therefore \sin x = -\frac{24}{25}; \quad \cos x = -\frac{7}{25}.$$

9. $\sec x = -\frac{17}{15}$, x being in the second quadrant.

$$\therefore \cos x = -\frac{15}{17}; \quad \sin x = +\frac{8}{17}; \quad \tan x = -\frac{8}{15}.$$

10. $\operatorname{cosec} x = -\frac{41}{5}$, x being in the fourth quadrant.

$$\therefore \sin x = -\frac{5}{41}; \quad \cos x = +\frac{40}{41}; \quad \tan x = -\frac{8}{40}.$$

11. If $\sin \frac{1}{2}\theta = \sqrt{\frac{(s-b)(s-c)}{bc}}$ where $s = \frac{a+b+c}{2}$, show that

$$\cos \frac{1}{2}\theta = \sqrt{\frac{s(s-a)}{bc}}.$$

12. If $\tan \frac{1}{2}\theta = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}$ where $s = \frac{a+b+c}{2}$, show that

$$\cos \frac{1}{2}\theta = \sqrt{\frac{s(s-c)}{ab}}.$$

13. If $\sec \theta = a$, show that $\sin \theta$ is imaginary if a is numerically less than unity.

$$\sin \theta = \sqrt{1 - \cos^2 \theta} = \sqrt{1 - \frac{1}{\sec^2 \theta}} = \sqrt{1 - \frac{1}{a^2}} = \frac{\sqrt{a^2 - 1}}{a}.$$

14. If $\tan \theta = a$, show that $\operatorname{cosec} \theta$ is real for all values of a .

15. If $\cos \theta = a$, show that $\operatorname{cosec} \theta$ is imaginary when a is numerically greater than unity.

49. The Signs of the Functions are given by the formulas of Art. 46, so that it is necessary to remember only that the sine is positive in the first and second quadrants and the cosine in the first and fourth. Thus, in the second quadrant,

$$\begin{aligned}\tan x &= \frac{\sin x}{\cos x} = \frac{+}{-} = -; \cot x = \frac{\cos x}{\sin x} = \frac{-}{+} = -; \\ \sec x &= \frac{1}{\cos x} = \frac{+}{-} = -; \operatorname{cosec} x = \frac{1}{\sin x} = \frac{+}{+} = +.\end{aligned}$$

50. Find the Values of the Following Expressions :

1. $\frac{\operatorname{vers} x \tan x - 1}{\sec x}$ when $\tan x = 4$, x being in the third quadrant. Find the numerical values of $\operatorname{vers} x$ and $\sec x$, and substitute.

$$\therefore \cos x = -\frac{1}{\sqrt{17}}, \sec x = -\sqrt{17}, \operatorname{vers} x = 1 + \frac{1}{\sqrt{17}}.$$

$$\therefore \frac{\frac{\sqrt{17}+1}{\sqrt{17}} \cdot 4 - 1}{-\sqrt{17}} = \frac{4\sqrt{17}+4-\sqrt{17}}{-17} = -\frac{3\sqrt{17}+4}{17}.$$

2. $\frac{\sin x \sec x}{\cos x \operatorname{cosec} x}$ when $\operatorname{vers} x = \frac{3}{4}$, x in the fourth quadrant. *Ans.* + 15.

3. $\frac{\tan x - \cot x}{\tan x + \cot x}$ when $\operatorname{cosec} x = -\sqrt{5}$, x in the third quadrant. *Ans.* $-\frac{3}{5}$.

4. $\frac{\sec x + \sin x}{\operatorname{cosec} x + \cos x}$ when $\cot x = -\frac{1}{2}$, x in the second quadrant. *Ans.* - 2.

5. $\frac{\sin x + \tan x}{\cos x + \operatorname{vers} x}$ when $\sec x = -\frac{4}{3}$, x in the third quadrant. *Ans.* $+\frac{1}{3}$.

6. $\frac{\sec x - \operatorname{vers} x}{\sec x + \operatorname{vers} x}$ when $\cot x = -2$, x in the second quadrant.

$$\text{Ans. } \frac{9+2\sqrt{5}}{1-2\sqrt{5}} = -\frac{29+20\sqrt{5}}{19}.$$

7. $\frac{\sin x + \tan^2 x}{\cos^2 x + \operatorname{vers}^2 x}$ when $\sec x = -\frac{4}{3}$, x in the second quadrant. *Ans.* $\frac{44}{133}$.

8. $\frac{\sec x + \sin x}{1 - \cot x}$ when $\tan x = 2$, x in the third quadrant. *Ans.* $-\frac{1}{5}\sqrt{5}$.

9. $\frac{\operatorname{cosec} x + \sec x}{\cot x \cos x}$ when $\sec x = +\sqrt{10}$, x in the fourth quadrant. *Ans.* - 20.
10. $\frac{\sec x - \operatorname{cosec} x}{\sec x + \operatorname{cosec} x}$ when $\cot x = -2$, x in the second quadrant. *Ans.* - 3.
11. $\frac{\operatorname{vers} x - \operatorname{covers} x}{\sec x - \operatorname{cosec} x}$ when $\sin x = -\frac{1}{3}$, x in the fourth quadrant. *Ans.* $-\frac{1}{3}\sqrt{5}$.

51. Change the Given Expression to Another containing only One Function :

1. $\frac{2 \sec^2 x + \sec^2 x \tan^2 x - \sec^4 x}{\sec^2 x - 1}$ to contain only cosec x .

It is best generally to change the expression to another containing only $\sin x$ and $\cos x$, and then to change this into one containing the proper function.

$$\begin{aligned} \therefore \frac{\frac{2}{\cos^2 x} + \frac{\sin^2 x}{\cos^4 x} - \frac{1}{\cos^4 x}}{\frac{1}{\cos^2 x} - 1} &= \frac{2 \cos^2 x + \sin^2 x - 1}{\cos^2 x(1 - \cos^2 x)} \\ &= \frac{2 - 2 \sin^2 x + \sin^2 x - 1}{(1 - \sin^2 x) \sin^2 x} = \frac{1 - \sin^2 x}{(1 - \sin^2 x) \sin^2 x} = \frac{1}{\sin^2 x} = \operatorname{cosec}^2 x. \end{aligned}$$

2. $\frac{\sin^2 x - \cos^2 x}{\operatorname{vers} x - \operatorname{covers} x}$ to contain only $\tan x$.

$$\therefore \frac{\sin^2 x - \cos^2 x}{1 - \cos x - 1 + \sin x} = \sin x + \cos x = \pm \frac{\tan x}{\sqrt{1 + \tan^2 x}} \pm \frac{1}{\sqrt{1 + \tan^2 x}},$$

where the signs used will depend upon the quadrant of x .

The true result is $\pm \frac{1 + \tan x}{\sqrt{1 + \tan^2 x}}$, where the positive sign corresponds to x in the first or fourth quadrant, and the negative to x in the second or third.

Use radicals as little as possible.

3. $1 - 2(1 - \operatorname{covers} x)^2 + \frac{\tan^4 x}{(1 + \tan^2 x)^2}$ to contain only $\cos x$. *Ans.* $\cos^4 x$.
4. $\frac{\sec x \operatorname{cosec} x - 4 \sin x \cos x}{\sin x \sec x}$ to contain only $\sin x$. *Ans.* $\frac{(1 - 2 \sin^2 x)^2}{\sin^2 x}$.
5. $\frac{(1 - \operatorname{covers} x)^2 \operatorname{cosec}^4 x}{(\operatorname{cosec}^2 x - 1) \cot^2 x}$ to contain only $\tan x$. *Ans.* $\tan^2 x + \tan^4 x$.
6. $\frac{\sec^2 x - \sec^2 x \sin^4 x (1 + \cot^2 x)}{\sin^2 x \cos^2 x}$ to contain only cosec x .

$$\text{Ans. } \frac{\operatorname{cosec}^4 x}{\operatorname{cosec}^2 x - 1}.$$

7. $\tan^2 \theta \sec^2 \theta - \sin^2 \theta \cos^2 \theta$ to contain only $\cot \theta$. *Ans.* $\frac{1+3 \cot^2 \theta + 3 \cot^4 \theta}{\cot^4 \theta (1 + \cot^2 \theta)^2}$.
8. $\frac{(1 - \tan^2 x)^2}{(1 + \tan^2 x)^2} (\cos^4 x - \sin^4 x)$ to contain only $\sin x$. *Ans.* $(1 - 2 \sin^2 x)^3$.
9. $\frac{\sec^2 a \sin^2 a}{(\tan a + 2 \cot a)^2}$ to contain only $\operatorname{cosec} a$. *Ans.* $\frac{1}{(2 \operatorname{cosec}^2 a - 1)^2}$.
10. $\frac{\sin^2 \theta \tan^2 \theta}{\sin^2 \theta - \cos^2 \theta}$ to contain only $\sec \theta$. *Ans.* $\frac{(\sec^2 \theta - 1)^2}{\sec^2 \theta - 2}$.
11. $\frac{\sec^2 \theta \operatorname{cosec}^2 \theta + \sec^2 \theta - \operatorname{cosec}^2 \theta - 1}{\tan^2 \theta - \operatorname{cosec}^2 \theta + 1}$ to contain only $\cot \theta$. *Ans.* $\frac{\cot^2 \theta + 2}{1 - \cot^4 \theta}$.

52. Solution of Trigonometric Equations.—Transform the given equation into one containing only a single function (usually the sine or cosine), because in a single equation we must have only one unknown quantity. Then solve the equation algebraically for this function as the unknown quantity. The corresponding angle may then be found from the tables. Test the angles by substitution in the given equation.

1. $\sin \theta \cos \theta = +\frac{1}{2}$.

$\therefore \sin \theta \sqrt{1 - \sin^2 \theta} = +\frac{1}{2}; \therefore \sin^2 \theta (1 - \sin^2 \theta) = \frac{1}{4};$

$\therefore \sin^4 \theta - \sin^2 \theta + \frac{1}{4} = 0; \therefore \sin^2 \theta - \frac{1}{2} = 0; \therefore \sin \theta = \pm \sqrt{\frac{1}{2}}.$

$\therefore \theta$ might be $45^\circ, 135^\circ, 225^\circ$, or 315° . But the given equation shows that the product of the sine and cosine must be positive, and hence that they must have the same sign. Both the sine and cosine are positive in the first quadrant, and negative in the third, but they have contrary signs in the second and fourth quadrants. Hence the only admissible values of θ are 45° and 225° .

2. $\tan \theta \sec \theta = -\sqrt{2}$.

$\therefore \theta = 225^\circ, 315^\circ.$

3. $\operatorname{cosec} \theta = \frac{2}{3} \tan \theta$.

$\therefore \theta = 60^\circ, 300^\circ.$

4. $\tan \theta + \cot \theta = 2$.

$\therefore \theta = 45^\circ, 225^\circ.$

5. $\sec^2 \theta + \operatorname{cosec}^2 \theta = 4$.

$\therefore \theta = 45^\circ, 135^\circ, 225^\circ, 315^\circ.$

6. $\sin \theta = \pm \sqrt{3}$ vers θ .

$\therefore \theta = 0^\circ, 60^\circ, 300^\circ.$

7. $\sec \theta + \tan \theta = \pm \sqrt{3}$.

$\therefore \theta = 30^\circ, 150^\circ. \quad [300^\circ, 330^\circ.]$

8. $\sec^2 \theta + \cot^2 \theta = \frac{1}{3}$.

$\therefore \theta = 30^\circ, 60^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ,$

9. $\sin x = +\sqrt{3} \cos x$.

$\therefore x = 60^\circ, 240^\circ.$

10. $\tan x = -2\sqrt{3} \cos x$.

$\therefore x = 240^\circ, 300^\circ.$

11. $\sin x \cos x = -\frac{1}{4}\sqrt{3}$.

$\therefore x = 120^\circ, 150^\circ, 300^\circ, 330^\circ.$

12. $\sin \theta + \operatorname{cosec} \theta = -\frac{1}{2}$.

$\therefore \theta = 210^\circ, 330^\circ.$

13. $3 \sin x = 2 \cos^2 x$.

$\therefore x = 30^\circ, 150^\circ.$

14. $\sec x \tan x = +2\sqrt{3}$.

$\therefore x = 60^\circ, 120^\circ.$

$$15. \sec \theta \text{ vers } \theta = 1 - \tan \theta.$$

$$\therefore \frac{1}{\cos \theta} (1 - \cos \theta) = 1 - \frac{\sin \theta}{\cos \theta}; \therefore \sin \theta = 2 \cos \theta - 1;$$

$$\therefore \sin^2 \theta = 4 \cos^2 \theta - 4 \cos \theta + 1; \therefore 1 - \cos^2 \theta = 4 \cos^2 \theta - 4 \cos \theta + 1;$$

$$\therefore 5 \cos^2 \theta - 4 \cos \theta = 0; \therefore \cos \theta (5 \cos \theta - 4) = 0;$$

$$\therefore \cos \theta = 0 \text{ and } 5 \cos \theta - 4 = 0.$$

(a) $\cos \theta = 0$ gives $\theta = 90^\circ$ or 270° . These values are rejected for reasons involving the methods of the Differential Calculus.

(b) $\cos \theta = \frac{4}{5}$ gives $\sin \theta = \pm \frac{3}{5}$, since this value of the cosine will allow the angle to lie either in the first or in the fourth quadrant. Transposing in the original equation, we have

$$\sec \theta \text{ vers } \theta + \tan \theta - 1 = 0,$$

and we test by substitution. For θ in the first quadrant, we have

$$\frac{5}{4} \cdot \frac{1}{5} + \frac{3}{4} - 1 = 0,$$

showing that θ has a value in the first quadrant. For θ in the fourth quadrant, we have

$$\frac{5}{4} \cdot \frac{1}{5} - \frac{3}{4} - 1 = -\frac{3}{2},$$

not zero; and hence θ does not have a value in the fourth quadrant.

$$16. \sin x \tan x = -\frac{2}{3}. \therefore \sin x = \pm \frac{2}{3}, \cos x = -\frac{4}{3}; \text{ quadrants II. and III.}$$

$$17. \text{vers } x = 2 \text{ covers } x. \therefore \cos x = \frac{3}{5} \text{ or } -1; \text{ first quadrant, and } 180^\circ.$$

$$18. \sin x \tan x = 2 \cos x. \therefore \sin x = \pm \sqrt{\frac{2}{3}}, \cos x = \pm \sqrt{\frac{1}{3}}; \text{ four quadrants.}$$

$$19. \sec x \operatorname{cosec} x = -2. \therefore \sin x = \pm \frac{1}{2}\sqrt{2}, \cos x = \mp \frac{1}{2}\sqrt{2}; 135^\circ \text{ and } 315^\circ.$$

$$20. \cos x \cot x = -\frac{5}{6}. \therefore \sin x = -\frac{5}{6}; \text{ quadrants III. and IV.}$$

$$21. \sin x \cos x = -\frac{1}{2}. \therefore \sin x = \pm \frac{1}{2} \text{ or } \pm \frac{1}{2}; \text{ quadrants II. and IV.}$$

$$22. \tan x = -\frac{2}{\sqrt{5}} \cos x. \therefore \sin x = -\frac{2}{\sqrt{5}}; \text{ quadrants III. and IV.}$$

$$23. \sec x + \tan x = 2. \therefore \tan x = +\frac{3}{4}; \text{ first quadrant.}$$

$$24. \sec x \tan 2x (1 - 2 \cos x) = 0.$$

The values of x are found by placing each factor equal to zero, and solving the resulting equations. Hence we have

$$\sec x = 0, \tan 2x = 0, 1 - 2 \cos x = 0.$$

But $\sec x = 0$ is impossible; $\tan 2x = 0$ gives $2x = 0^\circ$ or 180° , and, using the general measures of the angles, $x = 0^\circ, 90^\circ, 180^\circ, 270^\circ$, the second and the last values being inadmissible. $1 - 2 \cos x = 0$ gives $\cos x = \frac{1}{2}$, and $x = 60^\circ$ or 300° .

25. $\tan \frac{1}{2}x = 0. \quad \therefore x = 0^\circ.$
 26. $\text{vers } 3x = 0. \quad \therefore x = 0^\circ, 120^\circ, 240^\circ.$
 27. $\sin x \cos x (1 + 2 \cos x) = 0. \quad \therefore x = 0^\circ, 90^\circ, 180^\circ, 270^\circ, 120^\circ, 240^\circ. [330^\circ.$
 28. $\cos 2x (3 - 4 \cos^2 x) = 0. \quad \therefore x = 45^\circ, 225^\circ, 135^\circ, 315^\circ, 30^\circ, 150^\circ, 210^\circ,$
 29. $(1 + \tan x)(1 - 2 \sin x) = 0. \quad \therefore x = 45^\circ, 225^\circ, 30^\circ, 150^\circ.$
 30. $\tan x = -2 \sin x. \quad \therefore x = 0^\circ, 120^\circ, 180^\circ, 240^\circ.$
 31. $\sin 2x \text{ vers } 3x = 0. \quad \therefore x = 0^\circ, 90^\circ, 180^\circ, 120^\circ, 240^\circ, 270^\circ.$

53. The Functions of an Angle Greater than 360° are the same as those of the angle less than 360° , found by increasing or diminishing the given angle by some multiple of 360° ; for the position of the terminal side would not be changed by these operations. Thus

$$\tan 1010^\circ = \tan (1010^\circ - 720^\circ) = \tan 290^\circ;$$

$$\cos (-835^\circ) = \cos (-835^\circ + 720^\circ) = \cos (-115^\circ),$$

$$\text{or } \cos (-835^\circ) = \cos (-835^\circ + 1080^\circ) = \cos 245^\circ.$$

54. The Functions of $90^\circ \pm x$ and of $270^\circ \pm x$ are numerically equal to the cofunctions of x , but may differ from them in signs. Let the arcs EB, ED, KJ, KM , and NP be equal,

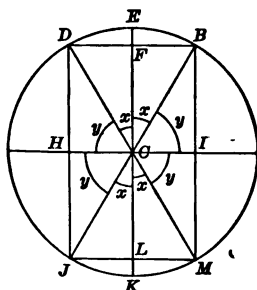


FIG. 36.

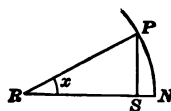


FIG. 37.

the radii CB and RP each being unity. Then the right triangles FCB, FCD, LCJ, LCM , and SRP are equal, having

the same hypotenuse (unity) and the angle x the same in each. Therefore

$$FB = CI = LM = DF = HC = JL = SP,$$

and $CF = IB = HD = LC = MI = JH = RS.$

$$\therefore \left. \begin{aligned} \sin(90^\circ - x) &= IB = CF = RS = +\cos x; \\ \cos(90^\circ - x) &= CI = FB = SP = +\sin x. \end{aligned} \right\} \quad (1)$$

$$\left. \begin{aligned} \sin(90^\circ + x) &= HD = CF = RS = +\cos x; \\ \cos(90^\circ + x) &= CH = FD = -DF^* = -SP = -\sin x. \end{aligned} \right\} \quad (2)$$

$$\left. \begin{aligned} \sin(270^\circ - x) &= HJ = CL = -LC^* = -RS = -\cos x; \\ \cos(270^\circ - x) &= CH = LJ = -JL^* = -SP = -\sin x. \end{aligned} \right\} \quad (3)$$

$$\left. \begin{aligned} \sin(270^\circ + x) &= IM = CL = -LC^* = -RS = -\cos x; \\ \cos(270^\circ + x) &= CI = LM = SP = +\sin x. \end{aligned} \right\} \quad (4)$$

$$\text{Thus } \sin 100^\circ = \sin(90^\circ + 10^\circ) = +\cos 10^\circ;$$

$$\cos 100^\circ = \cos(90^\circ + 10^\circ) = -\sin 10^\circ.$$

$$\sin 200^\circ = \sin(270^\circ - 70^\circ) = -\cos 70^\circ;$$

$$\cos 200^\circ = \cos(270^\circ - 70^\circ) = -\sin 70^\circ.$$

$$\sin 300^\circ = \sin(270^\circ + 30^\circ) = -\cos 30^\circ;$$

$$\cos 300^\circ = \cos(270^\circ + 30^\circ) = +\sin 30^\circ.$$

55. The Functions of $180^\circ \pm y$ and of $360^\circ - y$ are numerically equal to the *same* functions of y , but may differ from them in signs. From Fig. 36,

$$\left. \begin{aligned} \sin(180^\circ - y) &= HD = IB = +\sin y; \\ \cos(180^\circ - y) &= CH = -HC^* = -CI = -\cos y. \end{aligned} \right\} \quad (1)$$

$$\left. \begin{aligned} \sin(180^\circ + y) &= HJ = -JH^* = -IB = -\sin y; \\ \cos(180^\circ + y) &= CH = -HC^* = -CI = -\cos y. \end{aligned} \right\} \quad (2)$$

$$\left. \begin{aligned} \sin(360^\circ - y) &= IM = -MI^* = -IB = -\sin y; \\ \cos(360^\circ - y) &= CI = +\cos y. \end{aligned} \right\} \quad (3)$$

$$\text{Thus } \sin 100^\circ = \sin(180^\circ - 80^\circ) = +\sin 80^\circ;$$

$$\cos 100^\circ = \cos(180^\circ - 80^\circ) = -\cos 80^\circ.$$

$$\sin 200^\circ = \sin(180^\circ + 20^\circ) = -\sin 20^\circ;$$

$$\cos 200^\circ = \cos(180^\circ + 20^\circ) = -\cos 20^\circ.$$

$$\sin 300^\circ = \sin(360^\circ - 60^\circ) = -\sin 60^\circ;$$

$$\cos 300^\circ = \cos(360^\circ - 60^\circ) = +\cos 60^\circ.$$

* See Art. 2.

56. The Functions of a Negative Angle are numerically equal to the same functions of an equal positive angle, but may differ from them in signs.

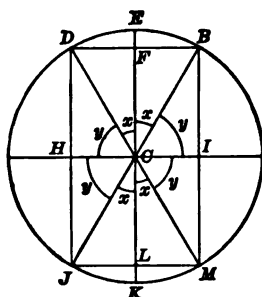


FIG. 88.

$$\left. \begin{aligned} \sin(-y) &= IM = -MI^* = -IB \\ &= -\sin y; \\ \cos(-y) &= CI = +\cos y. \end{aligned} \right\} (1)$$

Thus

$$\begin{aligned} \sin(x-180^\circ) &= \sin[-(180^\circ-x)] \\ &= -\sin(180^\circ-x) = -\sin x. \end{aligned}$$

$$\begin{aligned} \cos(x-180^\circ) &= \cos[-(180^\circ-x)] \\ &= +\cos(180^\circ-x) = -\cos x. \end{aligned}$$

57. Summary.—Using the equations of Art. 46,

$$\tan x = \frac{\sin x}{\cos x}, \cot x = \frac{\cos x}{\sin x}, \sec x = \frac{1}{\cos x}, \operatorname{cosec} x = \frac{1}{\sin x},$$

and the results of Arts. 54, 55, and 56, we have

$$\left. \begin{aligned} \sin(90^\circ - x) &= +\cos x; & \cos(90^\circ - x) &= +\sin x; \\ \tan(90^\circ - x) &= +\cot x; & \cot(90^\circ - x) &= +\tan x; \\ \sec(90^\circ - x) &= +\operatorname{cosec} x; & \operatorname{cosec}(90^\circ - x) &= +\sec x. \end{aligned} \right\} (1)$$

$$\left. \begin{aligned} \sin(90^\circ + x) &= +\cos x; & \cos(90^\circ + x) &= -\sin x; \\ \tan(90^\circ + x) &= -\cot x; & \cot(90^\circ + x) &= -\tan x; \\ \sec(90^\circ + x) &= -\operatorname{cosec} x; & \operatorname{cosec}(90^\circ + x) &= +\sec x. \end{aligned} \right\} (2)$$

$$\left. \begin{aligned} \sin(180^\circ - x) &= +\sin x; & \cos(180^\circ - x) &= -\cos x; \\ \tan(180^\circ - x) &= -\tan x; & \cot(180^\circ - x) &= -\cot x; \\ \sec(180^\circ - x) &= -\sec x; & \operatorname{cosec}(180^\circ - x) &= +\operatorname{cosec} x. \end{aligned} \right\} (3)$$

$$\left. \begin{aligned} \sin(180^\circ + x) &= -\sin x; & \cos(180^\circ + x) &= -\cos x; \\ \tan(180^\circ + x) &= +\tan x; & \cot(180^\circ + x) &= +\cot x; \\ \sec(180^\circ + x) &= -\sec x; & \operatorname{cosec}(180^\circ + x) &= -\operatorname{cosec} x. \end{aligned} \right\} (4)$$

$$\left. \begin{aligned} \sin(270^\circ - x) &= -\cos x; & \cos(270^\circ - x) &= -\sin x; \\ \tan(270^\circ - x) &= +\cot x; & \cot(270^\circ - x) &= +\tan x; \\ \sec(270^\circ - x) &= -\operatorname{cosec} x; & \operatorname{cosec}(270^\circ - x) &= -\sec x. \end{aligned} \right\} (5)$$

$$\left. \begin{aligned} \sin(270^\circ + x) &= -\cos x; & \cos(270^\circ + x) &= +\sin x; \\ \tan(270^\circ + x) &= -\cot x; & \cot(270^\circ + x) &= -\tan x; \\ \sec(270^\circ + x) &= +\operatorname{cosec} x; & \operatorname{cosec}(270^\circ + x) &= -\sec x. \end{aligned} \right\} (6)$$

$$\left. \begin{aligned} \sin(360^\circ - x) &= -\sin x; & \cos(360^\circ - x) &= +\cos x; \\ \tan(360^\circ - x) &= -\tan x; & \cot(360^\circ - x) &= -\cot x; \\ \sec(360^\circ - x) &= +\sec x; & \operatorname{cosec}(360^\circ - x) &= -\operatorname{cosec} x. \end{aligned} \right\} (7)$$

$$\left. \begin{aligned} \sin(-x) &= -\sin x; & \cos(-x) &= +\cos x; \\ \tan(-x) &= -\tan x; & \cot(-x) &= -\cot x; \\ \sec(-x) &= +\sec x; & \operatorname{cosec}(-x) &= -\operatorname{cosec} x. \end{aligned} \right\} (8)$$

* See Art. 2.

These formulas may be remembered from the three facts :

(a) Whenever the angle is $90^\circ \pm x$, or $270^\circ \pm x$, the functions of the angle are numerically equal to the corresponding cofunctions of x .

(b) Whenever the angle is $180^\circ \pm x$, $360^\circ - x$, or $-x$, the functions of the angle are numerically equal to the *same* functions of x .

(c) The sign to be placed before the function of x is that of the original function when x is less than 90° . Thus

$$\sin(270^\circ + x) = -\cos x,$$

since, when $x < 90^\circ$, $270^\circ + x$ will be in the fourth quadrant, and $\sin(270^\circ + x)$ will therefore be negative.

58. General Method of Proof.—In Arts. 54, 55, and 56, both x and y were less than 90° , but the formulas in Art. 57 are true for all values of x . Suppose, for example, that we wish to

prove the formulas for $270^\circ + x$ when x is in the fourth quadrant, that is, when x is between 270° and 360° . Let $\angle KAE GJ = x$; then $\angle AEGJKAEGJ = 270^\circ + x$. Let $\angle AEGKJ' = x$. Then in the right triangles JCL and $J'CL'$ the angles JCL and $J'CL'$ are equal, each being $360^\circ - x$; therefore the triangles are equal, and $CL = CL'$ and $LJ = L'J'$ numerically. Algebraically $CL = -CL'$ and $LJ = +L'J'$.

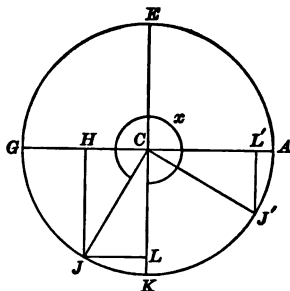


FIG. 39.

$$\begin{aligned}\therefore \sin(270^\circ + x) &= HJ = CL = -CL' = -\cos x; \\ \cos(270^\circ + x) &= CH = LJ = +L'J' = +\sin x.\end{aligned}$$

EXAMPLES.

1. From the preceding equations prove that

- | | |
|--|--|
| (a) $\tan(-1200^\circ) = \cot 30^\circ$. | (g) $\sin(-3000^\circ) = -\cos 30^\circ$. |
| (b) $\sec 1000^\circ = \operatorname{cosec} 10^\circ$. | (h) $\cos 1300^\circ = -\cos 40^\circ$. |
| (c) $\cos(-890^\circ) = -\cos 10^\circ$. | (i) $\tan 3200^\circ = -\tan 40^\circ$. |
| (d) $\cot 1700^\circ = \cot 80^\circ$. | (j) $\cot(-1300^\circ) = -\cot 40^\circ$. |
| (e) $\operatorname{cosec}(-1235^\circ) = -\sec 65^\circ$. | (k) $\sec(-2900^\circ) = +\sec 20^\circ$. |
| (f) $\sin 1340^\circ = -\cos 10^\circ$. | (l) $\operatorname{cosec} 2420^\circ = -\sec 10^\circ$. |

3. If $\tan \theta = -\cot 140^\circ$, find the two values of θ less than 360°

$$\therefore \tan \theta = -\cot (90^\circ + 50^\circ) = +\tan 50^\circ. \quad \therefore \theta = 50^\circ, 230^\circ.$$

3. Find the values of θ in the following equations:

- | | | |
|-----|---|--|
| (a) | $\sin \theta = +\cos 220^\circ$ | $\therefore 230^\circ, 310^\circ$ |
| (b) | $\sin \theta = +\cos 310^\circ$ | $\therefore 40^\circ, 140^\circ$ |
| (c) | $\sin \theta = -\cos 210^\circ$ | $\therefore 60^\circ, 120^\circ$ |
| (d) | $\sin^2 \theta = +\cos^2 200^\circ$ | $\therefore 70^\circ, 110^\circ, 250^\circ, 290^\circ$ |
| (e) | $\cos \theta = +\sin 150^\circ$ | $\therefore 60^\circ, 300^\circ$ |
| (f) | $\cos \theta = +\sin 250^\circ$ | $\therefore 160^\circ, 200^\circ$ |
| (g) | $\cos \theta = -\sin 170^\circ$ | $\therefore 100^\circ, 260^\circ$ |
| (h) | $\cos \theta = -\sin 275^\circ$ | $\therefore 5^\circ, 355^\circ$ |
| (i) | $\cos^2 \theta = +\sin^2 100^\circ$ | $\therefore 10^\circ, 350^\circ, 170^\circ, 190^\circ$ |
| (j) | $\tan \theta = +\cot 100^\circ$ | $\therefore 170^\circ, 350^\circ$ |
| (k) | $\tan \theta = +\cot 200^\circ$ | $\therefore 70^\circ, 250^\circ$ |
| (l) | $\tan \theta = -\cot 230^\circ$ | $\therefore 140^\circ, 320^\circ$ |
| (m) | $\cot \theta = +\tan 260^\circ$ | $\therefore 10^\circ, 190^\circ$ |
| (n) | $\cot \theta = +\tan 345^\circ$ | $\therefore 105^\circ, 285^\circ$ |
| (o) | $\cot \theta = -\tan 245^\circ$ | $\therefore 155^\circ, 335^\circ$ |
| (p) | $\cot \theta = -\tan 305^\circ$ | $\therefore 35^\circ, 215^\circ$ |
| (q) | $\sec \theta = -\operatorname{cosec} 100^\circ$ | $\therefore 170^\circ, 190^\circ$ |
| (r) | $\sec \theta = +\operatorname{cosec} 130^\circ$ | $\therefore 40^\circ, 320^\circ$ |
| (s) | $\sec \theta = +\operatorname{cosec} 310^\circ$ | $\therefore 140^\circ, 220^\circ$ |
| (t) | $\operatorname{cosec} \theta = +\sec 315^\circ$ | $\therefore 45^\circ, 135^\circ$ |
| (u) | $\operatorname{cosec} \theta = +\sec 230^\circ$ | $\therefore 220^\circ, 320^\circ$ |
| (v) | $\operatorname{cosec} \theta = -\sec 185^\circ$ | $\therefore 85^\circ, 95^\circ$ |
| (w) | $\operatorname{cosec} \theta = -\sec 335^\circ$ | $\therefore 245^\circ, 295^\circ$ |
| (x) | $\operatorname{cosec}^2 \theta = +\sec^2 250^\circ$ | $\therefore 20^\circ, 160^\circ, 200^\circ, 340^\circ$ |
| (y) | $\sec \theta = -\operatorname{cosec} 290^\circ$ | $\therefore 20^\circ, 340^\circ$ |
| (z) | $\sin \theta = -\cos 300^\circ$ | $\therefore 210^\circ, 330^\circ$ |

4. $\cos \theta = \sin 2\theta$. Show that one value of θ is 30° .

5. $\tan n\theta = -\cot 120^\circ$. Show that one value of θ is $30^\circ + n$.

6. $\sec 3\theta = \operatorname{cosec} (n-1)\theta$. Show that one value of θ is $90^\circ + (n+2)$.

7. If $\cot 309^\circ = -\frac{1}{\sqrt{5}}$, find $\sin 219^\circ$.

$$\begin{aligned} \sin 219^\circ &= \sin (180^\circ + 39^\circ) = -\sin 39^\circ. \quad \text{But } \cot 309^\circ = \cot (270^\circ + 39^\circ) = \\ &= -\tan 39^\circ; \quad \therefore \tan 39^\circ = +\frac{1}{\sqrt{5}}. \quad \therefore \sin 39^\circ = \frac{\tan 39^\circ}{\sqrt{1+\tan^2 39^\circ}} = \frac{1}{\sqrt{1+5}} = \frac{1}{\sqrt{6}}; \\ \therefore \sin 219^\circ &= -\frac{1}{\sqrt{6}} \end{aligned}$$

8. If $\sin 217^\circ = -\frac{1}{\sqrt{5}}$, prove that $\tan 127^\circ = -\frac{1}{2}$.

9. If $\cos 125^\circ = -a$, prove that $\tan 325^\circ = -\frac{a}{\sqrt{1-a^2}}$.

10. If $\cot 260^\circ = +a$, prove that $\cos 350^\circ = +\frac{1}{\sqrt{1+a^2}}$.

11. If $\sec 340^\circ = +a$, prove that $\sin 110^\circ = \frac{1}{a}$, and $\tan 110^\circ = -\frac{1}{\sqrt{a^2-1}}$.

12. If $\cos 300^\circ = +a$, prove that $\cot 120^\circ = -\frac{a}{\sqrt{1-a^2}}$.

13. If $\sin 115^\circ = +a$, prove that $\frac{\tan 205^\circ \sec 245^\circ}{\operatorname{cosec} 335^\circ} = +\frac{\sqrt{1-a^2}}{a}$.

14. If $\cos 200^\circ = -m$, prove that $\tan 110^\circ \operatorname{cosec} 250^\circ \cot 290^\circ = -\frac{1}{m}$.

15. If $\operatorname{cosec} 185^\circ = -m$, prove that $\tan 355^\circ \tan 275^\circ \cos 175^\circ = -\frac{\sqrt{m^2-1}}{m}$.

16. Show that $\cot \frac{1}{2}(-x-540^\circ) = \tan \frac{1}{2}x$.

$$\begin{aligned}\cot \frac{1}{2}(-x-540^\circ) &= \cot(-\frac{1}{2}x-90^\circ) \\ &= \cot[-(90^\circ + \frac{1}{2}x)] = -\cot(90^\circ + \frac{1}{2}x) = +\tan \frac{1}{2}x.\end{aligned}$$

17. Show that $\sin(y-90^\circ) = -\cos y$.

18. Show that $\sin(y-180^\circ) = -\sin y$.

19. Show that $\cos(y-270^\circ) = -\sin y$.

20. Show that $\sec(-x-540^\circ) = -\sec x$.

21. Show that $\tan(y-360^\circ) = +\tan y$.

22. Show that $\cos \frac{1}{2}(x-270^\circ) = +\sin \frac{1}{2}x$. [Note that 270° in the parenthesis is to be multiplied by $\frac{1}{2}$.]

23. Show that $\cos \frac{1}{2}(-810^\circ + a-b) = -\sin \frac{1}{2}(a-b)$.

24. Show that $\operatorname{cosec} \frac{1}{2}(x-360^\circ) = -\sec \frac{1}{2}x$.

25. Show that $\sec \frac{1}{2}(-900^\circ - x) = -\sec \frac{1}{2}x$.

26. Show that $\tan \frac{1}{2}(360^\circ + a-b) = +\tan \frac{1}{2}(a-b)$.

27. Show that $\cos(180^\circ - x)$ is equal to the sine of the complementary angle.

Complement $= 90^\circ - (180^\circ - x) = -(90^\circ - x)$; $\sin [-(90^\circ - x)] = -\sin(90^\circ - x) = -\cos x$. But $\cos(180^\circ - x) = -\cos x$. $\therefore \cos(180^\circ - x) = \sin[90^\circ - (180^\circ - x)]$. Q. E. D.

28. Show that $\operatorname{cosec}(270^\circ - x)$ equals the secant of the complementary angle.

29. Show that $\tan(180^\circ + x)$ equals the cotangent of the complementary angle.

30. Show that $\sec(270^\circ + x)$ equals the cosecant of the complementary angle.

31. Show that $\cos(90^\circ + x)$ equals the sine of the complementary angle.

32. Show that $\cot(360^\circ - x)$ equals the tangent of the complementary angle.

33. Show that $\tan(270^\circ + x)$ is equal to the negative of the tangent of the supplementary angle.

Supplement $= 180^\circ - (270^\circ + x) = -(90^\circ + x)$; $\tan [-(90^\circ + x)] = -\tan(90^\circ + x) = +\cot x$. But $\tan(270^\circ + x) = -\cot x$. $\therefore \tan(270^\circ + x) = -\tan[180^\circ - (270^\circ + x)]$. Q. E. D.

34. Show that $\operatorname{cosec}(180^\circ + x)$ is equal to the cosecant of the supplementary angle.

35. Show that $\sin(360^\circ - x)$ is equal to the sine of the supplementary angle.

36. Show that $\sec(90^\circ + x)$ is equal to the negative of the secant of the supplementary angle.

37. Show that $\cos(270^\circ - x)$ is equal to the negative of the cosine of the supplementary angle.

38. Show that $\cot(270^\circ + x)$ is equal to the negative of the cotangent of the supplementary angle.

59. The Trigonometric Tables.—The relations shown in Arts. 53 and 57 enable us to find the functions of any angle, although the tables contain only the sines, cosines, tangents, and cotangents of angles less than 45° . For, since

$$\sin(90^\circ - x) = \cos x, \quad \cos(90^\circ - x) = \sin x,$$

$$\tan(90^\circ - x) = \cot x, \quad \cot(90^\circ - x) = \tan x,$$

the tables are immediately extended to 90° by writing the proper degrees and minutes at the bottom and on the right of the page respectively.

Then, since the value of any function of an angle greater than 90° can be found in terms of a function of an angle less than 90° , we can find the numerical value of the function from the tables.

1. Find from the tables the logarithmic functions of $580^\circ 42'.4$.

$$580^\circ 42'.4 = 360^\circ + 220^\circ 42'.4.$$

$$\therefore \sin 580^\circ 42'.4 = \sin 220^\circ 42'.4 = \sin(180^\circ + 40^\circ 42'.4) = -\sin 40^\circ 42'.4;$$

$$\therefore \log \sin 580^\circ 42'.4 = 9.81437 n.$$

$$\cos 580^\circ 42'.4 = -\cos 40^\circ 42'.4; \therefore \log \cos 580^\circ 42'.4 = 9.87971 n.$$

$$\tan 580^\circ 42'.4 = +\tan 40^\circ 42'.4; \therefore \log \tan 580^\circ 42'.4 = 9.93467.$$

$$\cot 580^\circ 42'.4 = +\cot 40^\circ 42'.4; \therefore \log \cot 580^\circ 42'.4 = 0.06533.$$

2. Find from the tables the logarithmic functions of the following angles:

Angle.	log sin.	log cos.	log tan.	log cot.
$499^\circ 29'.7$.	9.81258.	9.88102 n.	9.93158 n.	0.06842 n.
$597^\circ 8'.3$.	9.92427 n.	9.73449 n.	0.18978.	9.81022.
$689^\circ 27'.6$.	9.70598 n.	9.93514.	9.77084 n.	0.22916 n.

3. $\sin b = \tan 250^\circ 15'.5 \cot 278^\circ 17'.3$; find $b = 203^\circ 57'.0$ or $336^\circ 3'.0$.

4. $\cos \beta = \cos 149^\circ 27'.6 \sin 216^\circ 44'.0$; find $\beta = 58^\circ 59'.7$ or $301^\circ 0'.3$.

5. $\tan \alpha = \sin 319^\circ 52'.0 + \cot 254^\circ 30'.2$; find $\alpha = 113^\circ 16'.5$ or $293^\circ 16'.5$.

6. $\cot c = \cos 216^\circ 44'.0 + \tan 329^\circ 27'.6$; find $c = 36^\circ 21'.6$ or $216^\circ 21'.6$.

60. Transform the First Member into the Second in the following examples. Usually it is best to change the given expression into one containing the sine and cosine, and then to change this into the required form. Any operation is admissible that does not change the *value* of the expression. Use radicals only when unavoidable. If the expression is factored, it is often advantageous to reduce each factor separately, not multiplying until it becomes necessary.

$$1. \frac{\tan x - \sin x}{\sin^3 x} = \frac{\sec x}{1 + \cos x}.$$

$$\begin{aligned} \frac{\tan x - \sin x}{\sin^3 x} &= \frac{\frac{\sin x}{\cos x} - \sin x}{\sin^3 x} = \frac{\sin x(1 - \cos x)}{\cos x \sin^3 x} = \frac{1 - \cos x}{\cos x \sin^2 x} \\ &= \frac{1 - \cos x}{\cos x(1 - \cos^2 x)} = \frac{1}{\cos x(1 + \cos x)} = \frac{\sec x}{1 + \cos x}. \end{aligned}$$

$$2. \cos x \tan x + \sin x \cot x = \sin x + \cos x.$$

$$3. (2 - \text{vers } x) \text{vers } x = \sin^2 x.$$

$$4. \frac{\cos x}{\sin x \cot^2 x} = \tan x.$$

$$5. (\tan x + \cot x) \sin x \cos x = 1.$$

$$6. (\sec^2 x - 1)(\text{cosec}^2 x - 1) = 1.$$

$$7. \sec x \text{cosec } x (\cos^2 x - \sin^2 x) = \cot x - \tan x.$$

$$8. (\sin x + \cos x)(\tan x + \cot x) = \sec x + \text{cosec } x.$$

$$9. \cot x + \frac{\sin x}{1 + \cos x} = \text{cosec } x.$$

$$10. \sin x (\sec x + \text{cosec } x) - \cos x (\sec x - \text{cosec } x) = \sec x \text{cosec } x.$$

$$11. (\text{cosec } x - \cot x)^2 = \frac{1 - \cos x}{1 + \cos x}.$$

$$12. (1 + \tan^2 x)(1 - \cot^2 x) = \sec^2 x - \text{cosec}^2 x.$$

$$13. \frac{\tan x - \cot x}{\tan x + \cot x} = \frac{2}{\text{cosec}^2 x} - 1. \quad [\text{First change to an expression containing}$$

only $\sin x$, the reciprocal of $\text{cosec } x].$

14. $\sec^2 x \text{cosec}^2 x - 2 = \tan^2 x + \cot^2 x.$ [Substitute for $\sec x$ and $\text{cosec } x$ their values in terms of $\tan x$ and $\cot x$ respectively.]

$$15. \frac{\tan \alpha + \tan \beta}{\cot \alpha + \cot \beta} = \tan \alpha \tan \beta.$$

16. $\cot x - \sec x \text{cosec } x (1 - 2 \sin^2 x) = \tan x.$ [The expression reduces to $\sin x + \cos x].$

17. $\operatorname{cosec} x (\sec x - 1) - \cot x (1 - \cos x) = \tan x - \sin x$. [Factor as soon as possible, and reduce each factor separately.]

$$18. \operatorname{vers} x (\sec x + 1) + \operatorname{covers} x (\operatorname{cosec} x + 1) = \sin x \tan x + \cos x \cot x.$$

$$19. \frac{\operatorname{vers} x (1 + \sec x)}{\sin x} + \frac{\operatorname{covers} x (1 + \operatorname{cosec} x)}{\cos x} = \sec x \operatorname{cosec} x.$$

$$20. \sin^2 x (\tan^2 x - 1) + \cos^2 x (\cot^2 x - 1) = \frac{(1 - 2 \cos^2 x)^2 \sec^4 x}{\tan^2 x}.$$

$$21. \sec x \operatorname{cosec} x [\operatorname{vers} x (\operatorname{vers} x - 2) - \operatorname{covers} x (\operatorname{covers} x - 2)] = \cot x - \tan x.$$

$$22. \cos^4 x - \sin^4 x = \cos x (1 - \tan x) (\sin x + \cos x).$$

23. $\frac{\operatorname{vers} x (1 + \cos x) - \operatorname{covers} x (1 + \sin x)}{\sec^2 x \operatorname{cosec}^2 x} = \frac{\tan^4 x - \tan^2 x}{\sec^6 x}$. [Change to an expression containing only $\sin x$ and $\cos x$, and then substitute their values in terms of $\tan x$.]

$$24. \frac{\sec^2 x \sin^2 x - \operatorname{cosec}^2 x + \operatorname{cosec}^2 x \cos^2 x}{\sec^2 x \sin^2 x - \operatorname{cosec}^2 x \cos^2 x} = \sin^2 x.$$

$$25. \tan^2 x - \sin^2 x \cos^2 x = \frac{(\sec^2 x + 1)(\sec^2 x - 1)^2}{\sec^4 x}.$$

$$26. \frac{\cos x \cot x - \sin x \tan x}{\operatorname{cosec} x - \sec x} = 1 + \sin x \cos x.$$

$$27. \frac{(\sec x + \operatorname{cosec} x)^2}{\tan x + \cot x} = \frac{(1 + \tan x)^2}{\tan x}.$$

$$28. 2 + \frac{\sin^4 x + \cos^4 x}{\sin^2 x \cos^2 x} = \sec^2 x + \operatorname{cosec}^2 x.$$

$$29. \frac{\sec x + \operatorname{cosec} x}{\sec x - \operatorname{cosec} x} = \frac{\tan x + 1}{\tan x - 1} = \frac{1 + \cot x}{1 - \cot x}.$$

$$30. \frac{\sin x - \tan^2 x \operatorname{covers} x}{\operatorname{cosec} x \cot^2 x} = \frac{\sin^4 x}{(1 + \sin x) \cos^2 x}.$$

$$31. \sec^2 x \operatorname{cosec}^2 x [\operatorname{vers} x (\operatorname{vers} x - 2) - \operatorname{covers} x (\operatorname{covers} x - 2)] = \cot^2 x - \tan^2 x.$$

32. $\tan x + \cot x = \frac{\sec^2 x + \operatorname{cosec}^2 x}{\sec x \operatorname{cosec} x}$. [It is admissible to divide both numerator and denominator by $\sin^2 x \cos^2 x$.]

$$33. \tan^2 \alpha \tan^2 \beta - 1 = \frac{\sin^2 \alpha - \cos^2 \beta}{\cos^2 \alpha \cos^2 \beta} = \frac{\sin^2 \beta - \cos^2 \alpha}{\cos^2 \alpha \cos^2 \beta}.$$

$$34. \frac{1 - \tan^2 \alpha \tan^2 \beta}{\tan^2 \alpha \tan^2 \beta} = \frac{\cos^2 \alpha - \sin^2 \beta}{\sin^2 \alpha \sin^2 \beta} = \frac{\cos^2 \beta - \sin^2 \alpha}{\sin^2 \alpha \sin^2 \beta}.$$

$$35. \sin^2 x \tan^2 x + \cos^2 x \cot^2 x = \tan^2 x + \cot^2 x - 1.$$

$$36. \sin^2 x \tan x + \cos^2 x \cot x + 2 \sin x \cos x = \sec x \operatorname{cosec} x.$$

37. $\sec^4 x + \tan^4 x = 1 + 2 \sec^2 x \tan^2 x$. [It is admissible to add and subtract $2 \sec^2 x \tan^2 x$.]

$$38. (r \cos \phi)^2 + (r \sin \phi \sin \theta)^2 + (r \sin \phi \cos \theta)^2 = r^2.$$

$$\therefore r^2 \cos^2 \phi + r^2 \sin^2 \phi (\sin^2 \theta + \cos^2 \theta) = r^2 (\cos^2 \phi + \sin^2 \phi) = r^2, \\ \text{since } \sin^2 x + \cos^2 x = 1.$$

$$39. (2r \sin \alpha \cos \alpha)^2 + r^2 (\cos^2 \alpha - \sin^2 \alpha)^2 = r^2.$$

$$40. (a \sin \gamma)^2 + (a \cos \gamma \sin \delta)^2 + (a \cos \gamma \cos \delta)^2 = a^2.$$

$$41. (\cos a \cos b - \sin a \sin b)^2 + (\sin a \cos b + \cos a \sin b)^2 = 1.$$

$$42. (\cos a \cos b + \sin a \sin b)^2 + (\sin a \cos b - \cos a \sin b)^2 = 1.$$

$$43. (x \cos \theta - y \sin \theta)^2 + (x \sin \theta + y \cos \theta)^2 = x^2 + y^2.$$

$$44. \frac{1}{(\cos^2 x - \sin^2 x)^2} - \frac{4 \tan^2 x}{(1 - \tan^2 x)^2} = 1.$$

$$45. \frac{1}{4 \sin^2 x \cos^2 x} - \frac{(1 - \tan^2 x)^2}{4 \tan^2 x} = 1.$$

$$46. (3 \sin \alpha \cos^2 \alpha - \sin^3 \alpha)^2 + (\cos^3 \alpha - 3 \sin^2 \alpha \cos \alpha)^2 = 1.$$

$$47. x^2 + y^2 + z^2 = r^2 \text{ when}$$

$$x = r \cos \alpha \cos \beta + r \cos i \sin \alpha \sin \beta,$$

$$y = r \cos i \cos \alpha \sin \beta - r \sin \alpha \cos \beta,$$

$$z = r \sin i \sin \beta.$$

CHAPTER V.

RELATIONS BETWEEN FUNCTIONS OF SEVERAL ANGLES.

61. Sine and Cosine of the Sum of Two Angles. — Let x and y be the angles, each, as well as their sum, being less than 90° .

QC is perpendicular to OP , BC and DQ are perpendicular to OA , and EC is parallel to OA , the radius of the circle being unity. Then

$$x + y = A O Q,$$

the angle $EQC = x$, $OC = \cos y$, and $CQ = \sin y$.

$$\begin{aligned} \sin(x + y) &= DQ = BC + EQ \\ &= OC \sin BOC + CQ \cos EQC \\ &= \cos y \sin x + \sin y \cos x, \end{aligned}$$

or $\sin(x + y) = \sin x \cos y + \cos x \sin y. \quad (1)$

$$\begin{aligned} \cos(x + y) &= OD = OB - EC = OC \cos BOC - CQ \sin EQC \\ &= \cos y \cos x - \sin y \sin x, \end{aligned}$$

or $\cos(x + y) = \cos x \cos y - \sin x \sin y. \quad (2)$

1. $\sin 90^\circ = \sin(60^\circ + 30^\circ) = \sin 60^\circ \cos 30^\circ + \cos 60^\circ \sin 30^\circ$

$$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{1}{2} = 1.$$

2. $\cos 90^\circ = \cos(60^\circ + 30^\circ) = \cos 60^\circ \cos 30^\circ - \sin 60^\circ \sin 30^\circ$

$$= \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2} \cdot \frac{1}{2} = 0.$$

3. If $\sin \alpha = \frac{4}{5}$, and $\sin \beta = \frac{1}{3}$, find $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ when $\alpha < 90^\circ$, and $\beta < 90^\circ$

Ans. $\sin(\alpha + \beta) = \frac{11}{15}$, $\cos(\alpha + \beta) = \frac{1}{15}$.

4. If $\tan \alpha = \frac{3}{4}$, and $\tan \beta = \frac{7}{24}$, find $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$ when $\alpha < 90^\circ$, and $\beta < 90^\circ$.

Ans. $\sin(\alpha + \beta) = \frac{1}{5}$, $\cos(\alpha + \beta) = \frac{3}{5}$.

NOTE.—At a point A the angle of elevation DAB to the top of a vertical wall is α , and the angle of depression CAD to its base is β . Find the height CB of the wall, the horizontal distance AD being a feet.

$$CB = CD + DB = a \tan \beta + a \tan \alpha$$

$$= a(\tan \alpha + \tan \beta). \quad (3)$$

$$= a \left(\frac{\sin \alpha}{\cos \alpha} + \frac{\sin \beta}{\cos \beta} \right)$$

$$= a \frac{\sin \alpha \cos \beta + \cos \alpha \sin \beta}{\cos \alpha \cos \beta}$$

$$= a \frac{\sin(\alpha + \beta)}{\cos \alpha \cos \beta}. \quad (4)$$

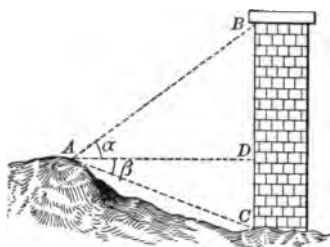


FIG. 41.

Eq. (3) would be solved by the use of the natural functions, while (4) is adapted to logarithmic computation.

62. Sine and Cosine of the Difference of Two Angles. — Let x and y be the angles, each being less than 90° and x being greater than y . QC is perpendicular to OP , BC and DQ are perpendicular to OA , and EQ is parallel to OA , the radius of the circle being unity. Then $x - y = \angle AOQ$, $\angle ECQ = x$, $OC = \cos y$, and $CQ = \sin y$.

$$\sin(x - y) = DQ = BC - EC$$

$$= OC \sin BOC - CQ \cos ECQ$$

$$= \cos y \sin x - \sin y \cos x,$$

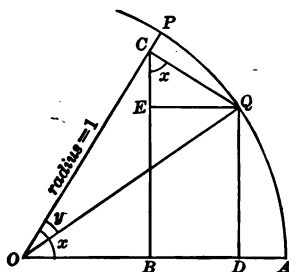


FIG. 42.

or $\sin(x - y) = \sin x \cos y - \cos x \sin y. \quad (1)$

$$\cos(x - y) = OD = OB + EQ = OC \cos BOC + CQ \sin ECQ$$

$$= \cos y \cos x + \sin y \sin x,$$

or $\cos(x - y) = \cos x \cos y + \sin x \sin y. \quad (2)$

In this proof we have assumed that x is the greater angle, but (1) and (2) are true when y is greater than x . To prove this, let β be greater than α . Then

$$\sin(\alpha - \beta) = \sin[-(\beta - \alpha)] = -\sin(\beta - \alpha),$$

and, developing $\sin(\beta - \alpha)$ by (1),

$$\begin{aligned}\sin(\alpha - \beta) &= -(\sin \beta \cos \alpha - \cos \beta \sin \alpha) \\ &= \sin \alpha \cos \beta - \cos \alpha \sin \beta.\end{aligned}\quad \text{Q.E.D.}$$

$$\begin{aligned}\text{Also } \cos(\alpha - \beta) &= \cos[-(\beta - \alpha)] = \cos(\beta - \alpha) \\ &= \cos \beta \cos \alpha + \sin \beta \sin \alpha.\end{aligned}\quad \text{Q.E.D.}$$

$$1. \sin 30^\circ = \sin(90^\circ - 60^\circ) = \sin 90^\circ \cos 60^\circ - \cos 90^\circ \sin 60^\circ$$

$$= 1 \cdot \frac{1}{2} - 0 \cdot \frac{\sqrt{3}}{2} = \frac{1}{2}.$$

$$2. \cos 30^\circ = \cos(90^\circ - 60^\circ) = \cos 90^\circ \cos 60^\circ + \sin 90^\circ \sin 60^\circ$$

$$= 0 \cdot \frac{1}{2} + 1 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}.$$

3. If $\sin \alpha = \frac{1}{11}$, and $\sin \beta = \frac{1}{11}$, find $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$ when $\alpha < 90^\circ$, and $\beta < 90^\circ$.

$$\text{Ans. } \sin(\alpha - \beta) = \frac{1}{11}; \cos(\alpha - \beta) = \frac{10}{11}.$$

4. If $\tan \alpha = \frac{1}{4}$, and $\tan \beta = \frac{1}{4}$, find $\sin(\alpha - \beta)$ and $\cos(\alpha - \beta)$ when $\alpha < 90^\circ$, and $\beta < 90^\circ$.

$$\text{Ans. } \sin(\alpha - \beta) = \frac{1}{15}; \cos(\alpha - \beta) = \frac{14}{15}.$$

NOTE.—At a point A on a horizontal plane, the angle CAD to the top of a crag is γ , and a feet farther away in the same vertical plane (at B), the angle CBD is γ' . Find $AC = x$.



FIG. 48.

$$CD = AC \tan \gamma = x \tan \gamma.$$

$$CD = BC \tan \gamma' = (a + x) \tan \gamma'.$$

$$\therefore x \tan \gamma = (a + x) \tan \gamma'.$$

$$\therefore x = \frac{a \tan \gamma'}{\tan \gamma - \tan \gamma'}.\quad (3)$$

$$= \frac{a \sin \gamma' \cos \gamma}{\sin \gamma \cos \gamma' - \cos \gamma \sin \gamma'} = \frac{a \sin \gamma' \cos \gamma}{\sin(\gamma - \gamma')}.\quad (4)$$

Eq. (3) would be solved by the use of the natural functions, while (4) is adapted to logarithmic computation.

63. General Proof of the Addition Formulas.—These formulas were shown in Art. 61 to be true when x , y , and $x + y$ were each less than 90° . That they are true for all values of these angles may be shown by proving the special cases separately. Let us consider first the case when

$$x < 90^\circ, y < 90^\circ, x + y > 90^\circ \text{ and } < 180^\circ.$$

Let $x = 90^\circ - \alpha$, $y = 90^\circ - \beta$; $\therefore x + y = 180^\circ - (\alpha + \beta)$.

$\therefore \alpha = 90^\circ - x$, $\beta = 90^\circ - y$, $\alpha + \beta = 180^\circ - (x + y)$.

$\therefore \alpha < 90^\circ$, $\beta < 90^\circ$, $\alpha + \beta < 90^\circ$, since $x + y > 90^\circ$.

Then $\sin(\alpha + \beta)$ may be developed by (1), Art. 61, since the conditions of that article are satisfied. But

$$\begin{aligned}\sin(x + y) &= \sin[180^\circ - (\alpha + \beta)] = \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ &= \sin(90^\circ - x) \cos(90^\circ - y) + \cos(90^\circ - x) \sin(90^\circ - y) \\ &= \cos x \sin y + \sin x \cos y. \quad \text{Q.E.D.}\end{aligned}$$

$$\begin{aligned}\text{Also } \cos(x + y) &= \cos[180^\circ - (\alpha + \beta)] = -\cos(\alpha + \beta) = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \\ &= -\cos(90^\circ - x) \cos(90^\circ - y) + \sin(90^\circ - x) \sin(90^\circ - y) \\ &= -\sin x \sin y + \cos x \cos y. \quad \text{Q.E.D.}\end{aligned}$$

Hence the formulas are true for $x < 90^\circ$, $y < 90^\circ$, $x + y < 180^\circ$.

To illustrate the proof for any special case, let us take x in the second and y in the fourth quadrant. Place $x = 90^\circ + \alpha$, and $y = 270^\circ + \beta$, so that α and β are each less than 90° . Then

$$\begin{aligned}\sin(x + y) &= \sin[360^\circ + (\alpha + \beta)] = \sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ &= \sin(x - 90^\circ) \cos(y - 270^\circ) + \cos(x - 90^\circ) \sin(y - 270^\circ) \\ &= (-\cos x)(-\sin y) + \sin x \cos y \\ &= \cos x \sin y + \sin x \cos y. \quad \text{Q.E.D.}\end{aligned}$$

Let the student prove the addition formulas in the following cases:

1. x in the first, and y in the third quadrant.
2. x in the second, and y in the second quadrant.
3. x in the second, and y in the third quadrant.
4. x in the third, and y in the third quadrant.
5. x in the third, and y in the fourth quadrant.
6. x in the fourth, and y in the fourth quadrant.

64. General Proof of the Subtraction Formulas.—These formulas were shown in Art. 62 to be true when x and y were each less than 90° , both for $x > y$ and for $x < y$. That they are true for all values of the angles may be shown by proving the special cases separately. For illustration, let x be in the second, and y in the third quadrant. Place $x = 90^\circ + \alpha$, and $y = 180^\circ + \beta$, so that $\alpha < 90^\circ$, and $\beta < 90^\circ$. Then

$$\begin{aligned}\sin(x - y) &= \sin[90^\circ + \alpha - (180^\circ + \beta)] = \sin[-90^\circ + (\alpha - \beta)] = -\sin[90^\circ - (\alpha - \beta)] \\ &= -\cos(\alpha - \beta) = -\cos \alpha \cos \beta - \sin \alpha \sin \beta \\ &= -\cos(x - 90^\circ) \cos(y - 180^\circ) - \sin(x - 90^\circ) \sin(y - 180^\circ) \\ &= -\sin x(-\cos y) - (-\cos x)(-\sin y) \\ &= \sin x \cos y - \cos x \sin y. \quad \text{Q.E.D.}\end{aligned}$$

$$\begin{aligned}\text{Also } \cos(x - y) &= \cos[-90^\circ + (\alpha - \beta)] = \cos[90^\circ - (\alpha - \beta)] = \sin(\alpha - \beta) \\ &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \\ &= \sin(x - 90^\circ) \cos(y - 180^\circ) - \cos(x - 90^\circ) \sin(y - 180^\circ) \\ &= (-\cos x)(-\cos y) - \sin x(-\sin y) \\ &= \cos x \cos y + \sin x \sin y. \quad \text{Q.E.D.}\end{aligned}$$

Let the student prove the subtraction formulas in the following cases :

1. x in the fourth, and y in the first quadrant.
2. x in the fourth, and y in the second quadrant.
3. x in the fourth, and y in the third quadrant.
4. x in the third, and y in the third quadrant.
5. x in the third, and y in the fourth quadrant.
6. x in the second, and y in the fourth quadrant.

65. Tangent of the Sum and of the Difference of Two Angles.

$$\tan(x+y) = \frac{\sin(x+y)}{\cos(x+y)} = \frac{\sin x \cos y + \cos x \sin y}{\cos x \cos y - \sin x \sin y}$$

Divide both numerator and denominator by $\cos x \cos y$.

$$\therefore \tan(x+y) = \frac{\frac{\sin x \cos y}{\cos x \cos y} + \frac{\cos x \sin y}{\cos x \cos y}}{\frac{\cos x \cos y}{\cos x \cos y} - \frac{\sin x \sin y}{\cos x \cos y}} = \frac{\frac{\sin x}{\cos x} + \frac{\sin y}{\cos y}}{1 - \frac{\sin x \sin y}{\cos x \cos y}};$$

$$\therefore \tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y} \quad (1)$$

In the same way, we may show that

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y} \quad (2)$$

$$1. \tan 75^\circ = \tan(45^\circ + 30^\circ) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ} = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = 2 + \sqrt{3}.$$

$$2. \tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ} = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}.$$

3. If $\sin \alpha = \frac{1}{2}$ and $\sin \beta = \frac{1}{3}$, find $\tan(\alpha + \beta)$ and $\tan(\alpha - \beta)$, when $\alpha < 90^\circ$ and $\beta < 90^\circ$.
Ans. $\tan(\alpha + \beta) = -\frac{1}{11}$; $\tan(\alpha - \beta) = \frac{1}{11}$.

66. Geometrical Proof.—In Fig. 44, let $OA = 1$, $AOB = x$, $BOC = y$. Draw BC perpendicular to OB , and CD parallel to OA ; $\therefore DBC = x$, $DCE = x + y$. Then

$$\tan(x+y) = AE = AB + BD + DE.$$

But $BD = BC \cos x = OB \tan y \cos x = \sec x \tan y \cos x = \tan y$,
 and $DE = CD \tan (x + y) = BC \sin x \tan (x + y) = OB \tan y \sin x \tan (x + y)$
 $= \sec x \tan y \sin x \tan (x + y) = \tan x \tan y \tan (x + y)$.
 $\therefore \tan (x + y) = \tan x + \tan y + \tan x \tan y \tan (x + y)$.

$$\therefore \tan (x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

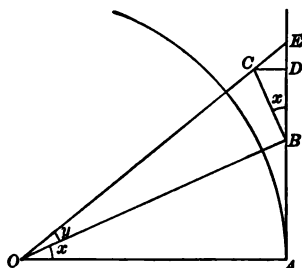


FIG. 44.

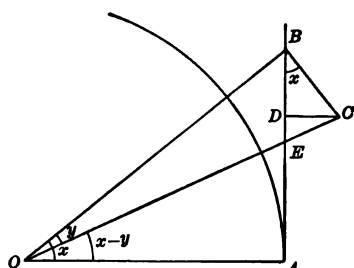


FIG. 45.

In Fig. 45, let $OA = 1$, $AOB = x$, $COB = y$. Draw BC perpendicular to OB , and DC parallel to OA ; $\therefore DBC = x$, $DCE = x - y$. Then

$$\tan (x - y) = AE = AB - DB - ED.$$

But $DB = BC \cos x = OB \tan y \cos x = \sec x \tan y \cos x = \tan y$,
 and $ED = DC \tan (x - y) = BC \sin x \tan (x - y) = OB \tan y \sin x \tan (x - y)$
 $= \sec x \tan y \sin x \tan (x - y) = \tan x \tan y \tan (x - y)$.
 $\therefore \tan (x - y) = \tan x - \tan y - \tan x \tan y \tan (x - y)$.

$$\therefore \tan (x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

EXAMPLES.

Find by inspection one value of x in Exs. (1-6):

- $\sin (n - 1)\alpha \cos \alpha + \cos (n - 1)\alpha \sin \alpha = \sin x$. *Ans.* $x = n\alpha$.
- $\cos (10^\circ + \alpha) \cos (10^\circ - \alpha) + \sin (10^\circ + \alpha) \sin (10^\circ - \alpha) = \cos x$. *Ans.* $x = 2\alpha$.
- $\sin (\alpha - \beta + 10^\circ) \cos (\beta - \alpha + 10^\circ) - \cos (\alpha - \beta + 10^\circ) \sin (\beta - \alpha + 10^\circ) = \sin x$.
Ans. $x = 2(\alpha - \beta)$.
- $\cos 45^\circ \cos (90^\circ - \alpha) - \sin 45^\circ \sin (90^\circ - \alpha) = \cos x$. *Ans.* $x = 135^\circ - \alpha$.
- $\sin (90^\circ + \frac{1}{2}\alpha) \cos (90^\circ - \frac{1}{2}\alpha) + \cos (90^\circ + \frac{1}{2}\alpha) \sin (90^\circ - \frac{1}{2}\alpha) = \sin x$.
Ans. $x = 180^\circ$.
- $\cos (45^\circ - \alpha) \cos (45^\circ + \alpha) - \sin (45^\circ - \alpha) \sin (45^\circ + \alpha) = \cos x$. *Ans.* $x = 90^\circ$.
- Given the functions of 30° and 45° , find those of 75° .

$$\text{Ans. } \sin 75^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}; \cos 75^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}; \tan 75^\circ = \frac{\sqrt{3} + 1}{\sqrt{3} - 1} = 2 + \sqrt{3}.$$

8. Given the functions of 30° and 45° , find those of 15° .

$$\text{Ans. } \sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}; \cos 15^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}; \tan 15^\circ = 2-\sqrt{3}.$$

9. If $\tan \alpha = \frac{4}{3}$ and $\sin \beta = \frac{1}{2}$, find the functions of $\alpha + \beta$ when α is in the third, and β in the second quadrant.

$$\text{Ans. } \sin(\alpha + \beta) = -\frac{7}{10}; \cos(\alpha + \beta) = \frac{24}{25}; \tan(\alpha + \beta) = -\frac{7}{24}.$$

10. If $\cos \alpha = -\frac{4}{5}$ and $\sin \beta = -\frac{1}{2}$, find the functions of $\alpha - \beta$ when α is in the third, and β in the fourth quadrant.

$$\text{Ans. } \sin(\alpha - \beta) = -\frac{7}{10}; \cos(\alpha - \beta) = -\frac{24}{25}; \tan(\alpha - \beta) = +\frac{7}{24}.$$

11. If $\cos \alpha = \frac{3}{5}$ and $\sin \beta = -\frac{1}{2}$, find the functions of $\alpha + \beta$ and of $\alpha - \beta$ when α is in the fourth, and β in the third quadrant.

$$\text{Ans. } \sin(\alpha + \beta) = +\frac{1}{5}; \cos(\alpha + \beta) = -\frac{7}{10}; \tan(\alpha + \beta) = -\frac{1}{7}; \\ \sin(\alpha - \beta) = +1; \cos(\alpha - \beta) = 0; \tan(\alpha - \beta) = \infty.$$

Transform the first member into the second (or last) in Exs. (12-32):

$$12. \sin(\alpha + \beta) \sin(\alpha - \beta) = \sin^2 \alpha - \sin^2 \beta = \cos^2 \beta - \cos^2 \alpha.$$

$$13. \cos(\alpha + \beta) \cos(\alpha - \beta) = \cos^2 \alpha - \sin^2 \beta = \cos^2 \beta - \sin^2 \alpha.$$

$$14. \sin(60^\circ + \alpha) - \sin \alpha = \sin(60^\circ - \alpha).$$

$$15. (r' \cos v' - r \cos v)^2 + (r' \sin v' - r \sin v)^2 = r^2 + r'^2 - 2rr' \cos(v' - v).$$

$$16. \cos^2 \alpha + \cos^2 \beta - 2 \cos \alpha \cos \beta \cos \omega = \sin^2 \omega, \text{ when } \omega = \alpha + \beta. \quad [\text{Place } \alpha = \omega - \beta.]$$

$$17. \tan \alpha + \frac{\tan \phi \sec \alpha}{\cos \alpha - \tan \phi \sin \alpha} = \tan(\alpha + \phi).$$

$$18. \sin^2 \theta + \sin^2(\omega - \theta) + 2 \sin \theta \cos \omega \sin(\omega - \theta) = \sin^2 \omega.$$

$$19. \cos^2 \theta + \cos^2(\omega - \theta) - 2 \cos \theta \cos \omega \cos(\omega - \theta) = \sin^2 \omega.$$

$$20. \tan x \pm \tan y = \frac{\sin(x \pm y)}{\cos x \cos y}.$$

$$21. \cot x \pm \cot y = \frac{\sin(y \pm x)}{\sin x \sin y}.$$

$$22. \cot x \pm \tan y = \frac{\cos(x \mp y)}{\sin x \cos y}.$$

$$23. \tan(30^\circ + x) + \tan(30^\circ - x) = \sin 60^\circ \sec(30^\circ + x) \sec(30^\circ - x).$$

$$24. \frac{1 - \tan \alpha}{1 + \tan \alpha} = \tan(45^\circ - \alpha). \quad [\text{Note that } 1 = \tan 45^\circ.]$$

$$25. \frac{1 - \cot \alpha}{1 + \cot \alpha} = -\tan(45^\circ - \alpha). \quad [\text{Note that } 1 = \cot 45^\circ.]$$

$$26. \sin(60^\circ + \alpha) - \sin(60^\circ - \alpha) = \sin \alpha.$$

$$27. \tan(45^\circ + \alpha) - \tan(45^\circ - \alpha) = \frac{4 \tan \alpha}{1 - \tan^2 \alpha}.$$

$$28. \frac{\sin(x + y)}{\cos(x - y)} = \frac{\tan x + \tan y}{1 + \tan x \tan y} = \frac{\cot x + \cot y}{1 + \cot x \cot y}.$$

$$\begin{aligned} 29. \sin(a+b+c) &= \sin[(a+b)+c] = \sin(a+b)\cos c + \cos(a+b)\sin c \\ &= (\sin a \cos b + \cos a \sin b)\cos c + (\cos a \cos b - \sin a \sin b)\sin c \\ &= \sin a \cos b \cos c + \cos a \sin b \cos c + \cos a \cos b \sin c - \sin a \sin b \sin c. \end{aligned}$$

$$\begin{aligned} 30. \cos(a+b+c) &= \cos a \cos b \cos c - \sin a \sin b \cos c - \sin a \cos b \sin c \\ &\quad - \cos a \sin b \sin c. \end{aligned}$$

$$31. \tan(a+b+c) = \frac{\tan a + \tan b + \tan c - \tan a \tan b \tan c}{1 - \tan a \tan b - \tan a \tan c - \tan b \tan c}.$$

$$\begin{aligned} 32. \sin(a+b-c) &= \sin a \cos b \cos c + \cos a \sin b \cos c - \cos a \cos b \sin c \\ &\quad + \sin a \sin b \sin c. \end{aligned}$$

33. If $\sin a = \frac{1}{2}$, $\sin \beta = -\frac{1}{2}$, $\sin \gamma = -\frac{1}{2}$, find $\sin(a - \beta - \gamma)$, when a , β , and γ are in the second, third, and fourth quadrants respectively. *Ans.* $-\frac{1}{2}$.

34. If $\sin a = \frac{1}{2}$, $\cos \beta = \frac{1}{2}$, $\tan \gamma = \frac{1}{2}$, find $\cos(a - \beta + \gamma)$, when a , β , and γ are in the second, fourth, and third quadrants respectively. *Ans.* $+\frac{1}{2}$.

67. To express the Functions of an Angle in Terms of those of Half the Angle.—In (1) and (2), Art. 61, let $y = x$.

$$\therefore \sin 2x = \sin x \cos x + \cos x \sin x = 2 \sin x \cos x. \quad (1)$$

$$\cos 2x = \cos^2 x - \sin^2 x, \quad (2)$$

$$\cos 2x = 1 - \sin^2 x - \sin^2 x = 1 - 2 \sin^2 x, \quad (3)$$

$$\text{or} \quad \cos 2x = \cos^2 x - 1 + \cos^2 x = 2 \cos^2 x - 1. \quad (4)$$

From (1), Art. 65,

$$\tan 2x = \frac{\tan x + \tan x}{1 - \tan x \tan x} = \frac{2 \tan x}{1 - \tan^2 x}. \quad (5)$$

1. To express the functions of 40° in terms of those of 20° , we have

$$\sin 40^\circ = 2 \sin 20^\circ \cos 20^\circ;$$

$$\cos 40^\circ = \cos^2 20^\circ - \sin^2 20^\circ;$$

$$\tan 40^\circ = \frac{2 \tan 20^\circ}{1 - \tan^2 20^\circ}.$$

2. $\sin \theta = \frac{2}{\sqrt{5}}$, θ being in the second quadrant. Find $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$.

$$\text{Ans. } \sin 2\theta = -\frac{4}{5}; \cos 2\theta = -\frac{3}{5}; \tan 2\theta = +\frac{4}{3}.$$

3. $\tan \theta = +2$, θ being in the third quadrant. Find $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$.

$$\text{Ans. } \sin 2\theta = +\frac{4}{5}; \cos 2\theta = -\frac{3}{5}; \tan 2\theta = -\frac{4}{3}.$$

4. $\cot \theta = -\frac{1}{2}$, θ being in the fourth quadrant. Find $\sin 2\theta$, $\cos 2\theta$, $\tan 2\theta$.

$$\text{Ans. } \sin 2\theta = -\frac{4}{5}; \cos 2\theta = -\frac{3}{5}; \tan 2\theta = +\frac{4}{3}.$$

NOTE.—To find the area of a right triangle, given c and a , we have

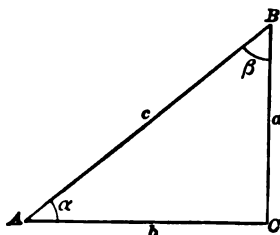


FIG. 46.

$$\text{area} = \frac{1}{2} ab.$$

But $a = c \sin \alpha$, and $b = c \cos \alpha$.

$$\therefore \text{area} = \frac{1}{2} c^2 \sin \alpha \cos \alpha; \quad (6)$$

$$\therefore \text{area} = \frac{1}{4} c^2 \sin 2\alpha. \quad (7)$$

In (6) we should have to find both $\sin \alpha$ and $\cos \alpha$ from the tables, in (7) we find only $\sin 2\alpha$, so that time is saved by using (7) instead of (6).

68. Geometrical Proof. — Let the radius OA of the circle be unity.

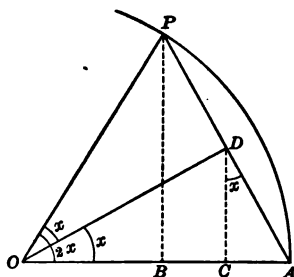


FIG. 47.

$$\begin{aligned} \sin 2x &= BP = 2 CD = 2 OD \sin x \\ &= 2 OA \cos x \sin x. \end{aligned}$$

$$\begin{aligned} \cos 2x &= OB = OC - BC = OC - CA \\ &= OD \cos x - AD \sin x \\ &= OA \cos^2 x - OA \sin^2 x \\ &= OA (\cos^2 x - \sin^2 x). \end{aligned}$$

$$\begin{aligned} \tan 2x &= \frac{BP}{OB} = \frac{2CD}{OC - CA} = \frac{2 OC \tan x}{OC - CD \tan x} \\ &= \frac{2 OC \tan x}{OC - OC \tan^2 x} = \frac{2 \tan x}{1 - \tan^2 x}. \end{aligned}$$

69. To express the Functions of an Angle in Terms of those of Double the Angle. — From Art. 67,

$$\cos 2x = 1 - 2 \sin^2 x;$$

$$\therefore 2 \sin^2 x = 1 - \cos 2x. \quad (1)$$

Also

$$\cos 2x = 2 \cos^2 x - 1;$$

$$\therefore 2 \cos^2 x = 1 + \cos 2x. \quad (2)$$

From (1) and (2), $\tan^2 x = \frac{1 - \cos 2x}{1 + \cos 2x}. \quad (3)$

$$\therefore \tan x = \pm \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x}}. \quad (4)$$

$$\therefore \tan x = \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x} \cdot \frac{1 + \cos 2x}{1 + \cos 2x}} = \sqrt{\frac{1 - \cos^2 2x}{(1 + \cos 2x)^2}};$$

$$\therefore \tan x = \frac{\sin 2x}{1 + \cos 2x}. \quad (5)$$

Also $\tan x = \sqrt{\frac{1 - \cos 2x}{1 + \cos 2x} \cdot \frac{1 - \cos 2x}{1 - \cos 2x}} = \sqrt{\frac{(1 - \cos 2x)^2}{1 - \cos^2 2x}};$

$$\therefore \tan x = \frac{1 - \cos 2x}{\sin 2x}. \quad (6)$$

NOTE. — The double sign is not used in (5) and (6), for

$$\frac{\sin 2x}{1 + \cos 2x} = \frac{2 \sin x \cos x}{2 \cos^2 x} = \tan x,$$

and

$$\frac{1 - \cos 2x}{\sin 2x} = \frac{2 \sin^2 x}{2 \sin x \cos x} = \tan x.$$

1. To express the functions of 20° in terms of those of 40° , we have

$$2 \sin^2 20^\circ = 1 - \cos 40^\circ;$$

$$2 \cos^2 20^\circ = 1 + \cos 40^\circ;$$

$$\tan^2 20^\circ = \frac{1 - \cos 40^\circ}{1 + \cos 40^\circ};$$

$$\tan 20^\circ = \frac{\sin 40^\circ}{1 + \cos 40^\circ} = \frac{1 - \cos 40^\circ}{\sin 40^\circ}.$$

2. $\tan 2\theta = -2$, 2θ being in the second quadrant. Find the functions of θ .

$$\therefore \cos 2\theta = -\frac{1}{\sqrt{5}}, \text{ and } \sin 2\theta = +\frac{2}{\sqrt{5}}.$$

$$\therefore \sin \theta = \pm \sqrt{\frac{1}{2} \left(1 + \frac{1}{\sqrt{5}} \right)}, \text{ from (1);}$$

$$\cos \theta = \pm \sqrt{\frac{1}{2} \left(1 - \frac{1}{\sqrt{5}} \right)}, \text{ from (2);}$$

$$\tan \theta = \frac{\frac{2}{\sqrt{5}}}{1 - \frac{1}{\sqrt{5}}} = \frac{2}{\sqrt{5} - 1}, \text{ from (5).}$$

Since 2θ is in the second quadrant, θ may be either in the first or in the third quadrant; hence $\sin \theta$ and $\cos \theta$ have the double sign, and $\tan \theta$ is positive.

3. Given the functions of 30° , find those of 15° .

$$\text{Ans. } \sin 15^\circ = \frac{\sqrt{3} - 1}{2\sqrt{2}}; \cos 15^\circ = \frac{\sqrt{3} + 1}{2\sqrt{2}}; \tan 15^\circ = 2 - \sqrt{3}.$$

4. Given the functions of 45° , find those of $22\frac{1}{2}^\circ$.

$$\text{Ans. } \sin 22\frac{1}{2}^\circ = \frac{1}{2}\sqrt{2 - \sqrt{2}}; \cos 22\frac{1}{2}^\circ = \frac{1}{2}\sqrt{2 + \sqrt{2}}; \tan 22\frac{1}{2}^\circ = \sqrt{2} - 1.$$

70. Geometrical Proof. — Let the radius CA of the circle be unity.

$$\sin x = \frac{BP}{OP} = \frac{\sqrt{OB \cdot BA}}{\sqrt{OB \cdot OA}} = \sqrt{\frac{BA}{OA}} = \sqrt{\frac{BA}{2}}$$

$$= \sqrt{\frac{CA - CB}{2}} = \sqrt{\frac{1 - \cos 2x}{2}}.$$

$$\cos x = \frac{OB}{OP} = \frac{OB}{\sqrt{OB \cdot OA}} = \sqrt{\frac{OB}{OA}} = \sqrt{\frac{OB}{2}}$$

$$= \sqrt{\frac{OC + CB}{2}} = \sqrt{\frac{1 + \cos 2x}{2}}.$$

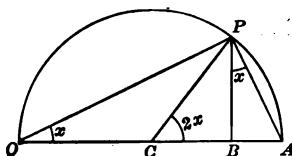


FIG. 43.

$$\tan x = \frac{BP}{OB} = \frac{BP}{OC + CB} = \frac{\sin 2x}{1 + \cos 2x}$$

$$\tan x = \frac{BA}{BP} = \frac{CA - CB}{BP} = \frac{1 - \cos 2x}{\sin 2x}$$

71. Multiple Angles.—Suppose that we wish to express $\sin 3x$ in terms of powers of $\sin x$.

$$\begin{aligned}\sin 3x &= \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x \\ &= 2 \sin x \cos^2 x + (1 - 2 \sin^2 x) \sin x \\ &= 2 \sin x - 2 \sin^3 x + \sin x - 2 \sin^3 x \\ &= 3 \sin x - 4 \sin^3 x. \quad \text{Q.E.D.}\end{aligned}$$

1. Show that $\cos 3x = 4 \cos^3 x - 3 \cos x$.

2. Show that $\tan 3x = \frac{3 \tan x - \tan^3 x}{1 - 3 \tan^2 x}$.

3. Show that $\sin 4x = 4 \sin x \cos x - 8 \sin^3 x \cos x$. [Use $4x = 2x + 2x$.]

4. Show that $\cos 4x = 1 - 8 \sin^2 x + 8 \sin^4 x$.

5. Show that $\tan 4x = \frac{4 \tan x (1 - \tan^2 x)}{1 - 6 \tan^2 x + \tan^4 x}$.

6. Show that $\sin 5x = 5 \sin x - 20 \sin^3 x + 16 \sin^5 x$. [Use $5x = 3x + 2x$.]

7. Show that $\cos 5x = 5 \cos x - 20 \cos^3 x + 16 \cos^5 x$.

8. Find the functions of 18° , of 36° , and of 72° .

Place $x = 18^\circ$; then, since $\cos 54^\circ = \sin 36^\circ$, we have

$$\cos 3x = \sin 2x.$$

$$\therefore 4 \cos^3 x - 3 \cos x = 2 \sin x \cos x.$$

$$\therefore \cos x (4 \cos^2 x - 3 - 2 \sin x) = 0.$$

$$\therefore 1 - 4 \sin^2 x - 2 \sin x = 0.$$

$$\therefore \sin x = \frac{1}{4}(-1 \pm \sqrt{5}).$$

$$\therefore \sin 18^\circ = \cos 72^\circ = \frac{1}{4}(\sqrt{5} - 1); \quad \cos 18^\circ = \sin 72^\circ = \frac{1}{4}\sqrt{10 + 2\sqrt{5}}.$$

Hence

$$\sin 36^\circ = 2 \sin 18^\circ \cos 18^\circ = \frac{1}{4}\sqrt{10 - 2\sqrt{5}};$$

$$\cos 36^\circ = 1 - 2 \sin^2 18^\circ = \frac{1}{4}(\sqrt{5} + 1).$$

72. To change the Product of Functions of Angles into the Sum of Functions.—From Arts. 61 and 62,

$$\sin(x + y) = \sin x \cos y + \cos x \sin y;$$

$$\sin(x - y) = \sin x \cos y - \cos x \sin y.$$

$$\therefore \sin(x + y) + \sin(x - y) = 2 \sin x \cos y, \quad (1)$$

and $\sin(x + y) - \sin(x - y) = 2 \cos x \sin y. \quad (2)$

Also $\cos(x + y) = \cos x \cos y - \sin x \sin y;$
 $\cos(x - y) = \cos x \cos y + \sin x \sin y.$
 $\therefore \cos(x + y) + \cos(x - y) = 2 \cos x \cos y,$ (3)
 and $\cos(x + y) - \cos(x - y) = -2 \sin x \sin y.$ (4)

Reversing (1), (2), (3), and (4), we have

$$\sin x \cos y = \frac{1}{2} \sin(x + y) + \frac{1}{2} \sin(x - y). \quad (5)$$

$$\cos x \sin y = \frac{1}{2} \sin(x + y) - \frac{1}{2} \sin(x - y). \quad (6)$$

$$\cos x \cos y = \frac{1}{2} \cos(x + y) + \frac{1}{2} \cos(x - y). \quad (7)$$

$$\sin x \sin y = -\frac{1}{2} \cos(x + y) + \frac{1}{2} \cos(x - y). \quad (8)$$

In applying these formulas, let x represent the larger angle.

1. $\sin 4\theta \cos 2\theta = \frac{1}{2} \sin(4\theta + 2\theta) + \frac{1}{2} \sin(4\theta - 2\theta)$, from (5),
 $= \frac{1}{2} \sin 6\theta + \frac{1}{2} \sin 2\theta.$
2. $\cos 6\theta \sin 2\theta = \frac{1}{2} \sin(6\theta + 2\theta) - \frac{1}{2} \sin(6\theta - 2\theta)$, from (6),
 $= \frac{1}{2} \sin 8\theta - \frac{1}{2} \sin 4\theta.$
3. $\cos 8\theta \cos 2\theta = \frac{1}{2} \cos 10\theta + \frac{1}{2} \cos 6\theta$, from (7).
4. $\sin 6\theta \sin 4\theta = -\frac{1}{2} \cos 10\theta + \frac{1}{2} \cos 2\theta$, from (8).
5. $\cos 2\theta \sin 4\theta = \frac{1}{2} \sin 6\theta - \frac{1}{2} \sin(-2\theta)$, from (6),
 $= \frac{1}{2} \sin 6\theta + \frac{1}{2} \sin 2\theta$, as in Ex. 1.
6. $\sin 2\theta \cos 6\theta = \frac{1}{2} \sin 8\theta + \frac{1}{2} \sin(-4\theta)$, from (5),
 $= \frac{1}{2} \sin 8\theta - \frac{1}{2} \sin 4\theta$, as in Ex. 2.
7. $\cos 2\theta \cos 8\theta = \frac{1}{2} \cos 10\theta + \frac{1}{2} \cos(-6\theta)$, from (7),
 $= \frac{1}{2} \cos 10\theta + \frac{1}{2} \cos 6\theta$, as in Ex. 3.
8. $\sin 4\theta \sin 6\theta = -\frac{1}{2} \cos 10\theta + \frac{1}{2} \cos(-2\theta)$, from (8),
 $= -\frac{1}{2} \cos 10\theta + \frac{1}{2} \cos 2\theta$, as in Ex. 4.
9. $\sin^2 \theta \cos \theta = \sin \theta [\sin \theta \cos \theta] = \sin \theta [\frac{1}{2} \sin(\theta + \theta) + \frac{1}{2} \sin(\theta - \theta)]$
 $= \sin \theta [\frac{1}{2} \sin 2\theta + \frac{1}{2} \sin 0^\circ] = \frac{1}{2} \sin \theta \sin 2\theta$
 $= \frac{1}{2} [-\frac{1}{2} \cos 3\theta + \frac{1}{2} \cos \theta] = -\frac{1}{4} \cos 3\theta + \frac{1}{4} \cos \theta.$
10. Reduce $\sin^3 \alpha \cos \alpha$ to $\frac{1}{4} \sin 2\alpha - \frac{1}{4} \sin 4\alpha.$

$$\sin^3 \alpha \cos \alpha = \sin^2 \alpha \cdot \sin \alpha \cos \alpha;$$

using (8) and (5), or the relations in Arts. 69 and 67, we have

$$\begin{aligned} \sin^3 \alpha \cos \alpha &= \frac{1}{2} (1 - \cos 2\alpha) \cdot \frac{1}{2} \sin 2\alpha = \frac{1}{4} \sin 2\alpha - \frac{1}{4} \sin 2\alpha \cos 2\alpha \\ &= \frac{1}{4} \sin 2\alpha - \frac{1}{4} \sin 4\alpha. \end{aligned}$$

11. Reduce $\sin^2 \theta \cos^2 \theta$ to $\frac{1}{8} (1 - \cos 4\theta).$
12. Reduce $\sin^2 \theta \cos^3 \theta$ to $\frac{1}{8} (\cos \theta - \frac{1}{2} \cos 3\theta - \frac{1}{2} \cos 5\theta).$
13. Reduce $\sin^3 \theta \cos^3 \theta$ to $\frac{1}{16} (3 \sin 2\theta - \sin 6\theta).$
14. Reduce $\cos^5 \theta$ to $\frac{1}{16} (10 \cos \theta + 5 \cos 3\theta + \cos 5\theta).$
15. Reduce $\cos^3 \theta$ to $\frac{1}{4} (\cos 3\theta + 3 \cos \theta).$
16. Reduce $\sin^5 \theta \cos^3 \theta$ to $\frac{1}{16} (3 \sin 2\theta - \sin 4\theta - \sin 6\theta + \frac{1}{2} \sin 8\theta).$

73. To change the Algebraic Sum of Functions of Angles into the Product of Functions.—Let $x + y = u$ and $(x - y) = v$.

$$\therefore x = \frac{1}{2}(u + v) \text{ and } y = \frac{1}{2}(u - v).$$

Substituting in (1), (2), (3), and (4), Art. 72, we have

$$\sin u + \sin v = 2 \sin \frac{1}{2}(u + v) \cos \frac{1}{2}(u - v). \quad (1)$$

$$\sin u - \sin v = 2 \cos \frac{1}{2}(u + v) \sin \frac{1}{2}(u - v). \quad (2)$$

$$\cos u + \cos v = 2 \cos \frac{1}{2}(u + v) \cos \frac{1}{2}(u - v). \quad (3)$$

$$\cos u - \cos v = -2 \sin \frac{1}{2}(u + v) \sin \frac{1}{2}(u - v). \quad (4)$$

In applying the formulas, let u represent the larger angle.

1. Reduce $\sin 3\theta + \sin \theta$ to $2 \sin 2\theta \cos \theta$.

Let $u = 3\theta$ and $v = \theta$ in (1).

2. Reduce $\cos \theta - \cos 3\theta$ to $4 \sin^2 \theta \cos \theta$.

$\cos \theta - \cos 3\theta = -(\cos 3\theta - \cos \theta)$. Let $u = 3\theta$ and $v = \theta$ in (4).

$$\begin{aligned} \therefore -(\cos 3\theta - \cos \theta) &= -(-2 \sin 2\theta \sin \theta) = +2 \sin 2\theta \sin \theta \\ &= 4 \sin \theta \cos \theta \sin \theta = 4 \sin^2 \theta \cos \theta. \end{aligned}$$

3. Reduce $\sin 3\theta + \cos \theta$ to a product.

$$\begin{aligned} \sin 3\theta + \cos \theta &= \sin 3\theta + \sin (90^\circ - \theta) = 2 \sin (45^\circ + \theta) \cos (2\theta - 45^\circ) \\ &= 2 \sin (45^\circ + \theta) \cos (45^\circ - 2\theta). \end{aligned}$$

74. Geometrical Proof.—In the figure, OD bisects the angle QOP , and is therefore perpendicular to QP . Using the notation there shown, we have

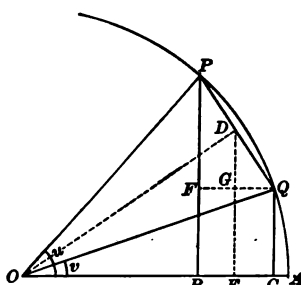


FIG. 49.

$QOP = u - v$; $\therefore QOD = DOP = \frac{1}{2}(u - v)$;
 $AOD = AOQ + QOD = v + \frac{1}{2}(u - v) = \frac{1}{2}(u + v)$;
 $FPQ = GDQ = AOD = \frac{1}{2}(u + v)$. Then, if the radius = 1,

$$\begin{aligned} \sin u + \sin v &= BP + CQ = 2 ED = 2 OD \sin AOD \\ &= 2 OP \cos DOP \sin AOD \\ &= 2 \sin \frac{1}{2}(u + v) \cos \frac{1}{2}(u - v). \quad (1) \end{aligned}$$

$$\begin{aligned} \sin u - \sin v &= BP - CQ = 2 GD = 2 DQ \cos GDQ \\ &= 2 OQ \sin QOD \cos GDQ \\ &= 2 \cos \frac{1}{2}(u + v) \sin \frac{1}{2}(u - v). \quad (2) \end{aligned}$$

$$\begin{aligned} \cos u + \cos v &= OB + OC = 2 OE = 2 OD \cos AOD = 2 OP \cos DOP \cos AOD \\ &= 2 \cos \frac{1}{2}(u + v) \cos \frac{1}{2}(u - v). \quad (3) \end{aligned}$$

$$\begin{aligned} \cos u - \cos v &= OB - OC = -2 GQ = -2 DQ \sin GDQ = -2 OQ \sin QOD \sin GDQ \\ &= -2 \sin \frac{1}{2}(u + v) \sin \frac{1}{2}(u - v). \quad (4) \end{aligned}$$

EXAMPLES.

Show that the first member of the equation may be reduced to the second (or last) in Exs. (1-7):

1. $\sin(45^\circ + x) + \sin(45^\circ - x) = 2 \sin 45^\circ \cos x = \sqrt{2} \cos x.$
2. $\sin(90^\circ + x) - \sin(180^\circ + x) = 2 \cos(135^\circ + x) \sin(-45^\circ)$
 $= -\sqrt{2} \cos(135^\circ + x).$
3. $\cos(180^\circ + x) + \cos(180^\circ - x) = 2 \cos 180^\circ \cos x = -2 \cos x.$
4. $\cos(270^\circ + x) - \cos(270^\circ - x) = -2 \sin 270^\circ \sin x = +2 \sin x.$
5. $\sin 3x + 2 \sin 5x + \sin 7x = 4 \sin 5x \cos^2 x.$
6. $\cos 3x + 2 \cos 5x + \cos 7x = 4 \cos 5x \cos^2 x.$
7. $\cos(b - c) - \cos a = +2 \sin \frac{1}{2}(a + b - c) \sin \frac{1}{2}(a - b + c).$
8. Show that $\sin(\lambda'' - \lambda') - \sin(\lambda'' - \lambda) + \sin(\lambda' - \lambda)$ may be reduced to $4 \sin \frac{1}{2}(\lambda' - \lambda) \sin \frac{1}{2}(\lambda'' - \lambda') \sin \frac{1}{2}(\lambda'' - \lambda).$ [The formula $\sin x = 2 \sin \frac{1}{2}x \cos \frac{1}{2}x$ is used in the process.]

If $\alpha + \beta + \gamma = 180^\circ$, reduce the first member to the second in Exs. (9-14):

9. $\sin \alpha + \sin \beta + \sin \gamma = 4 \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}\alpha \cos \frac{1}{2}\beta.$
 $\gamma = 180^\circ - (\alpha + \beta); \therefore \sin \gamma = \sin(\alpha + \beta).$ Then
 $\sin \alpha + \sin \beta + \sin(\alpha + \beta) = 2 \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha - \beta) + 2 \sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}(\alpha + \beta)$
 $= 2 \sin \frac{1}{2}(\alpha + \beta) [\cos \frac{1}{2}(\alpha - \beta) + \cos \frac{1}{2}(\alpha + \beta)]$
 $= 2 \sin \frac{1}{2}(\alpha + \beta) (2 \cos \frac{1}{2}\alpha \cos \frac{1}{2}\beta).$
10. $\cos \alpha + \cos \beta + \cos \gamma = 4 \cos \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta + 1.$ [Note that $\cos \gamma = -\cos(\alpha + \beta) = -2 \cos^2 \frac{1}{2}(\alpha + \beta) + 1.$]
11. $\cos 2\alpha + \cos 2\beta + \cos 2\gamma = -4 \cos \alpha \cos \beta \cos \gamma - 1.$
12. $\sin 2\alpha + \sin 2\beta + \sin 2\gamma = 4 \sin \alpha \sin \beta \sin \gamma.$
13. $2 \sin^2 \alpha + 2 \sin^2 \beta + 2 \sin^2 \gamma = 4 + 4 \cos \alpha \cos \beta \cos \gamma.$
14. $\sin 3\alpha + \sin 3\beta + \sin 3\gamma = -4 \cos \frac{3}{2}\alpha \cos \frac{3}{2}\beta \cos \frac{3}{2}\gamma.$
15. If $\alpha + \beta + \gamma = 360^\circ$, $\sin \alpha + \sin \beta + \sin \gamma = 4 \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta \sin \frac{1}{2}\gamma.$
16. If $\alpha + \beta + \gamma = 360^\circ$, $\sin \alpha + \sin \beta + 2 \sin \frac{1}{2}\gamma = 4 \sin \frac{1}{2}(\alpha + \beta) \cos^2 \frac{1}{2}(\alpha - \beta).$

75. Circular, or Inverse Trigonometric, Functions.—If y is the sine of the angle or arc x , then x is the arc whose sine is y . This is written $x = \sin^{-1} y$, read " x is the arc whose sine is y ." So also if $\tan x = m$, then " x is the arc whose tangent is m ," written $x = \tan^{-1} m$.

In consequence of this notation, if we have

$\frac{1}{\sin x}$ and wish to bring $\sin x$ into the numer-

ator, we must write it in a parenthesis with the exponent -1 ;

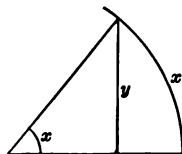


Fig. 50.

$\frac{1}{\sin x} = (\sin x)^{-1}$. All other exponents may be written above the name of the functions; $\frac{1}{\sin^2 x} = \sin^{-2} x = (\sin x)^{-2}$.

1. $y = \tan^{-1} m + \tan^{-1} n$. Find $\tan y$.

Let $\tan^{-1} m = a$ and $\tan^{-1} n = b$; $\therefore \tan a = m$, $\tan b = n$.

$$\therefore y = a + b; \therefore \tan y = \frac{\tan a + \tan b}{1 - \tan a \tan b} = \frac{m + n}{1 - mn}$$

2. $\tan^{-1} \frac{1}{m} = \tan^{-1} \frac{1}{m+n} + \tan^{-1} x$. Find x .

$$\therefore \tan^{-1} x = \tan^{-1} \frac{1}{m} - \tan^{-1} \frac{1}{m+n}. \text{ Let } a = \tan^{-1} \frac{1}{m}, b = \tan^{-1} \frac{1}{m+n}.$$

$$\therefore \tan^{-1} x = a - b; \therefore x = \tan(a - b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} = \frac{\frac{1}{m} - \frac{1}{m+n}}{1 + \frac{1}{m(m+n)}};$$

$$\therefore x = \frac{n}{m^2 + mn + 1}.$$

3.* $y = \sin^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{4}$. Find $\sin y$. *Ans.* $\sin y = \frac{1}{10}(4 + 3\sqrt{3})$.

4. $\tan^{-1} \frac{1}{m} = \tan^{-1} \frac{1}{m-n} - \tan^{-1} x$. Find $x = \frac{n}{m^2 - mn + 1}$.

5. $\tan^{-1} a = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$. Find $a = \frac{1}{2}$.

6.* $y = \sin^{-1} m + \sin^{-1} n$. Find $\sin y = m\sqrt{1-n^2} + n\sqrt{1-m^2}$.

7.* $y = \cos^{-1} m + \cos^{-1} n$. Find $\sin y = n\sqrt{1-m^2} + m\sqrt{1-n^2}$.

8.* $y = \cos^{-1} m - \sin^{-1} n$. Find $\cos y = m\sqrt{1-n^2} + n\sqrt{1-m^2}$.

9. $\tan^{-1} a = \tan^{-1} \frac{1}{2} - \tan^{-1} \frac{1}{3}$. Find $a = \frac{1}{2}$.

10.* $m = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$. Find $m = 45^\circ$.

11.* $m = \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{4} + \tan^{-1} \frac{1}{5}$. Find $m = 45^\circ$.

12.* $m = 2 \tan^{-1} \frac{1}{2} - \tan^{-1} \frac{1}{3}$. Find $m = 45^\circ$.

Let $\tan^{-1} \frac{1}{2} = a$, $\tan^{-1} \frac{1}{3} = b$; $\therefore m = 2a + b$; $\therefore \tan m =$, etc.

13.* $m = 2 \tan^{-1} \frac{1}{2} + \tan^{-1} \frac{1}{3}$. Find $m = 45^\circ$.

14.* Show that $\tan^{-1} \frac{1}{2}(1-m) = \sec^{-1} \frac{1}{2}\sqrt{5-2m+m^2}$.

Let $\tan^{-1} \frac{1}{2}(1-m) = x$; $\therefore \tan x = \frac{1}{2}(1-m)$; $\sec x = \sqrt{1+\tan^2 x}$;
 $\therefore \sec x = \frac{1}{2}\sqrt{5-2m+m^2}$. $\therefore x = \sec^{-1} \frac{1}{2}\sqrt{5-2m+m^2}$.

15. Show that $\tan^{-1} m = \frac{1}{2} \tan^{-1} \frac{2m}{1-m^2}$.

Let $x = \tan^{-1} m$, or $m = \tan x$. If the equation is true, we must have

$$x = \frac{1}{2} \tan^{-1} \frac{2m}{1-m^2}, \text{ or } 2x = \tan^{-1} \frac{2 \tan x}{1 - \tan^2 x}, \text{ or } \tan 2x = \frac{2 \tan x}{1 - \tan^2 x},$$

a formula proved in Art. 67.

16. Show that $\cos^{-1} m = \frac{1}{2} \cos^{-1}(2m^2 - 1)$.

17.* Show that $\sin^{-1} \frac{2\sqrt{ab}}{a+b} = \tan^{-1} \frac{2\sqrt{ab}}{a-b}$.

* When the angles are less than 90° .

18. Show that $\sin\left(\frac{\pi}{2} - 2 \tan^{-1} \sqrt{\frac{1-x}{1+x}}\right) = x$.

19.* Show that $\frac{1}{2} \text{vers}^{-1} \frac{1}{2} a^2 - \sin^{-1} \frac{1}{2} a$ is constant for all possible values of a .

Let $\theta = \frac{1}{2} \text{vers}^{-1} \frac{1}{2} a^2 - \sin^{-1} \frac{1}{2} a$, and let $m = \text{vers}^{-1} \frac{1}{2} a^2$, $n = \sin^{-1} \frac{1}{2} a$.

$$\therefore \theta = \frac{1}{2} m - n; \quad \therefore 2\theta = m - 2n; \quad \therefore \cos 2\theta = \cos m \cos 2n + \sin m \sin 2n.$$

But $\sin n = \frac{1}{2} a$; $\therefore \cos n = \frac{1}{2} \sqrt{4 - a^2}$;

$$\therefore \sin 2n = \frac{1}{2} a \sqrt{4 - a^2}; \quad \cos 2n = \frac{1}{2} (2 - a^2).$$

Also $\cos m = 1 - \text{vers } m = 1 - \frac{1}{2} a^2$; $\therefore \sin m = \frac{a}{2} \sqrt{4 - a^2}$.

$$\therefore \cos 2\theta = \frac{2 - a^2}{2} \cdot \frac{2 - a^2}{2} + \frac{a}{2} \sqrt{4 - a^2} \cdot \frac{a}{2} \sqrt{4 - a^2} = 1.$$

$$\therefore 2\theta = 0^\circ, \text{ or } \theta = 0^\circ.$$

20.* Show that $\tan^{-1} \frac{\sqrt{1-a^2}}{a} + \sin^{-1} a$ is constant for all possible values of a .

21.* Show that $\text{vers}^{-1} a - 2 \cot^{-1} \sqrt{\frac{2-a}{a}}$ is constant for all possible values of a .

22.* Show that $\text{vers}^{-1} \frac{x}{12} - 2 \sin^{-1} \sqrt{\frac{x}{24}}$ is constant for all possible values of x .

76. To prove that $\tan x > x > \sin x$ when $x < \frac{\pi}{2}$, x being expressed in Circular Measure. — Let $AOB = BOC = x$, the radius being unity. Evidently $AT > SB$, or $\tan x > \sin x$.

Also, since the shortest distance from a point to a line is perpendicular to the line, $SB < AB$, or $\sin x < x$.

The arc AC may be considered as composed of an infinite number of infinitesimal straight lines; hence $AT + TC > \text{arc } ABC$, since ABC is a convex polygon lying in the triangle formed by a chord AC with the tangent lines TA and TC . Then

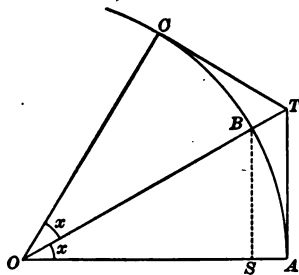


FIG. 51.

$$2 AT > \text{arc } ABC, \text{ or } AT > \text{arc } AB, \text{ or } \tan x > x.$$

Hence $\tan x > x$, and $x > \sin x$. Q.E.D.

77. To prove that $\sin x$, $\tan x$, and x approach Equality as the Angle x approaches Zero. — As the angle AOT decreases,

* When the angles are less than 90° .

the points B and T approach A , and hence approach each other. But

$$\frac{SB}{AT} = \frac{\sin x}{\tan x} = \cos x.$$

When the angle x approaches zero as its limit, $\cos x$ approaches unity as its limit. Hence $\frac{SB}{AT}$, or $\frac{\sin x}{\tan x}$, approaches unity as its limit, or $\sin x$ and $\tan x$ approach equality.

The arc x is intermediate in value between $\sin x$ and $\tan x$; hence the three quantities approach equality as the angle becomes smaller. That is, the three ratios

$$\frac{\sin x}{\tan x}, \quad \frac{\sin x}{x}, \quad \frac{\tan x}{x},$$

approach unity as the angle approaches zero.

Hence we may say that when the angle is small, its sine and its tangent are equal to the arc itself, and its cosine is equal to unity. The smaller the angle, the more nearly correct will be the assumption.

78. Development of $\sin x$, of $\cos x$, and of $\tan x$.—Let us assume that

$$\sin x = a + bx + cx^2 + dx^3 + ex^4 + fx^5 + \dots \quad (1)$$

is true for all values of x . Then it is true when x has the values $+y$ and $-y$; hence

$$\sin y = a + by + cy^2 + dy^3 + ey^4 + fy^5 + \dots \quad (2)$$

$$\text{and} \quad \sin(-y) = a - by + cy^2 - dy^3 + ey^4 - fy^5 + \dots \quad (3)$$

But $\sin y = -\sin(-y)$, or $\sin y + \sin(-y) = 0$. Adding (2) and (3),

$$2a + 2cy^2 + 2ey^4 + \dots = 0. \quad (4)$$

But (4) is true for all values of y , since (1) is true for all values of x . In order that all values of y may reduce the left member of (4) to zero, we must have $a = 0$, $c = 0$, $e = 0$, ... Hence (1) becomes

$$\sin x = bx + dx^3 + fx^5 + \dots \quad (5)$$

$$\text{or} \quad \frac{\sin x}{x} = b + dx^2 + fx^4 + \dots \quad (6)$$

But as x approaches zero, $\frac{\sin x}{x}$ approaches unity, and $b + dx^2 + fx^4 + \dots$ approaches b . Hence

$$1 = b, \quad (7)$$

and (5) becomes

$$\sin x = x + dx^3 + fx^5 + \dots \quad (8)$$

Again, let

$$\cos x = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \dots \quad (9)$$

Since $\cos x = \cos(-x)$, we have

$$\begin{aligned} & A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \dots \\ &= A - Bx + Cx^2 - Dx^3 + Ex^4 - Fx^5 + \dots \end{aligned} \quad (10)$$

$$\text{or} \quad 2Bx + 2Dx^3 + 2Fx^5 + \dots = 0. \quad (11)$$

In order that this may be true for all values of x , we must have $B = 0$, $D = 0$, $F = 0 \dots$, and (9) becomes

$$\cos x = A + Cx^2 + Ex^4 + \dots \quad (12)$$

But when $x = 0$, (12) reduces to

$$1 = A, \quad (13)$$

and hence (12) becomes

$$\cos x = 1 + Cx^2 + Ex^4 + \dots \quad (14)$$

Substituting from (14) and (8) in the formula

$$\cos 2x = \cos^2 x - \sin^2 x,$$

$$\begin{aligned} \text{we have} \quad 1 + 4Cx^2 + 16Ex^4 + \dots &= 1 + (2C - 1)x^2 \\ &+ (2E + C^2 - 2d)x^4 + \dots \end{aligned} \quad (15)$$

Equating the coefficients of like powers of x ,

$$4C = 2C - 1, \quad \text{or} \quad 2C + 1 = 0. \quad (16)$$

$$16E = 2E + C^2 - 2d, \quad \text{or} \quad 14E - C^2 + 2d = 0. \quad (17)$$

Substituting from (14) and (8) in the formula

$$\sin 2x = 2 \sin x \cos x,$$

$$\begin{aligned} \text{we have} \quad 2x + 8dx^3 + 32fx^5 + \dots &= 2x + 2(C + d)x^3 \\ &+ 2(E + Cd + f)x^5 + \dots \end{aligned} \quad (18)$$

Equating the coefficients of like powers of x ,

$$4d = C + d, \quad \text{or} \quad 3d - C = 0. \quad (19)$$

$$16f = E + Cd + f, \quad \text{or} \quad 15f - E - Cd = 0. \quad (20)$$

$$\text{From (16),} \quad C = -\frac{1}{3}. \quad (21)$$

$$\text{From (19),} \quad d = -\frac{1}{6} = -\frac{1}{\underline{3}}. \quad (22)$$

$$\text{From (17),} \quad E = +\frac{1}{24} = +\frac{1}{\underline{4}}. \quad (23)$$

$$\text{From (20),} \quad f = +\frac{1}{120} = +\frac{1}{\underline{5}}. \quad (24)$$

These values, substituted in (8) and (14), give

$$\sin x = x - \frac{x^3}{\underline{3}} + \frac{x^5}{\underline{5}} - \dots \quad (25)$$

$$\cos x = 1 - \frac{x^2}{\underline{2}} + \frac{x^4}{\underline{4}} - \dots \quad (26)$$

Dividing (25) by (26),

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots \quad (27)$$

In (25), (26), and (27), which are the required developments, x must be expressed in circular measure.

79. Computation of the Trigonometric Functions (First Method). — The functions may be computed by (25), (26), and (27), Art. 78. Thus, to find $\sin 20^\circ$, we place $x = \frac{1}{3}\pi$, the circular measure of 20° .

$\log \pi^3 = 1.49145$	$\log \pi^5 = 2.4857$	$x = \frac{\pi}{9} = 0.34906\ 59$
$\log 9^3 = 7.13727 - 10$	$\log 9^5 = 5.2288 - 10$	$\frac{x^3}{\underline{3}} = 0.00708\ 88$
$\log 6 = 9.22185 - 10$	$\log 120 = 7.9208 - 10$	$0.34197\ 71$
$\log \frac{x^3}{\underline{3}} = 7.85057 - 10$	$\log \frac{x^5}{\underline{5}} = 5.6353 - 10$	$\frac{x^5}{\underline{5}} = 0.00004\ 32$
$\therefore \frac{x^3}{\underline{3}} = 0.0070888$	$\therefore \frac{x^5}{\underline{5}} = 0.0000432$	$\therefore \sin 20^\circ = 0.34202\ 03$

In the tables, $\sin 20^\circ = 0.34202$.

80. Computation of the Trigonometric Functions (Second Method).—From (25), Art. 78, it may be shown that

$$\sin 1'' = 0.00000\ 48481\ 36811\ 07637,$$

while $\arcsin 1'' = 0.00000\ 48481\ 36811\ 09586.$

$$\therefore \arcsin 1'' - \sin 1'' = 0.00000\ 00000\ 00000\ 02.$$

Again, $\sin 1' = 0.00029\ 08882\ 04563\ 42460,$

while $\arcsin 1' = 0.00029\ 08882\ 08665\ 72160.$

$$\therefore \arcsin 1' - \sin 1' = 0.00000\ 00000\ 04.$$

Again, $\sin 1^\circ = 0.01745\ 24064\ 37283\ 51282,$

while $\arcsin 1^\circ = 0.01745\ 32925\ 19943\ 29577.$

$$\therefore \arcsin 1^\circ - \sin 1^\circ = 0.00000\ 09.$$

Also, from (26), Art. 78,

$$\cos 1'' = 0.99999\ 99999\ 88 = 1 - 0.00000\ 00000\ 12.$$

$$\cos 1' = 0.99999\ 99576\ 92 = 1 - 0.00000\ 00423\ 08.$$

$$\cos 1^\circ = 0.99984\ 76952 = 1 - 0.00015.$$

In computing a set of five-place tables, we may assume

$$\sin 1' = \arcsin 1' = 0.00029\ 08882 \text{ with an error of } 5 \times 10^{-12},$$

and $\cos 1' = 1$ with an error of $4 \times 10^{-8}.$

Then $\sin 2' = 2 \sin 1' \cos 1'; \cos 2' = \cos^2 1' - \sin^2 1'.$

$$\sin 3' = \sin 2' \cos 1' + \cos 2' \sin 1';$$

$$\cos 3' = \cos 2' \cos 1' - \sin 2' \sin 1'.$$

$$\sin 4' = \sin (3' + 1'); \cos 4' = \cos (3' + 1'),$$

or $\sin 4' = 2 \sin 2' \cos 2'; \cos 4' = \cos^2 2' - \sin^2 2'.$

And so on.

This method would be employed until the functions of all angles less than 30° had been computed. Then, since

$$\sin (30^\circ + x) = \cos x - \sin (30^\circ - x),$$

and $\cos (30^\circ + x) = \cos (30^\circ - x) - \sin x,$

the functions of angles between 30° and 45° would be found by combining the functions already found. Thus, if $x = 10^\circ$, we

have $\sin 40^\circ = \cos 10^\circ - \sin 20^\circ,$

and $\cos 40^\circ = \cos 20^\circ - \sin 10^\circ.$

It is possible to compute independently the sine and cosine of $3^\circ, 6^\circ, 9^\circ, \dots, 39^\circ, 42^\circ, 45^\circ$. We have found in this chapter* the sine and cosine of 15° , of 18° , and of 36° , and we have

$$\begin{aligned} 3^\circ &= 18^\circ - 15^\circ, & 6^\circ &= 36^\circ - 30^\circ, & 9^\circ &= 45^\circ - 36^\circ, & 12^\circ &= 30^\circ - 18^\circ, \\ 21^\circ &= 36^\circ - 15^\circ, & 24^\circ &= 45^\circ - 21^\circ, & 27^\circ &= 45^\circ - 18^\circ, \\ 33^\circ &= 18^\circ + 15^\circ, & 39^\circ &= 45^\circ - 6^\circ, & 42^\circ &= 45^\circ - 3^\circ. \end{aligned}$$

The values found from these relations would serve as checks upon the computation.

The computations may also be checked by Euler's and Legendre's verification formulas :

$$\sin(36^\circ + A) - \sin(36^\circ - A) - \sin(72^\circ + A) + \sin(72^\circ - A) = \sin A.$$

$$\cos(36^\circ + A) + \cos(36^\circ - A) - \cos(72^\circ + A) - \cos(72^\circ - A) = \cos A.$$

81. Approximate Assumptions. — It can be shown that

$$\tan 1'' - \text{arc } 1'' = 0.00000 \ 00000 \ 00000 \ 0\bar{4};$$

$$\text{arc } 1'' - \sin 1'' = 0.00000 \ 00000 \ 00000 \ 0\bar{2};$$

$$\tan 1'' - \sin 1'' = 0.00000 \ 00000 \ 00000 \ 0\bar{6}.$$

Hence we may assume that

$$\sin 1'' = \tan 1'' = \text{arc } 1''. \quad (1)$$

In the whole circumference of a circle there are $1296000''$, so that the error due to placing $\text{arc } 1'' = \sin 1''$ in finding the circumference of a circle with a radius of unity will be only $2\frac{1}{2}$ units in the eleventh decimal place.

In the computation of elliptic orbits there occurs the equation

$$M = E - e \sin E,$$

where M and E are expressed in circular measure. If M'' is the number of seconds in the angle, $M = M'' \text{ arc } 1''$, and approximately $M = M'' \sin 1''$ and $E = E'' \sin 1''$.

Hence the equation may be written

$$M'' = E'' - \frac{e}{\sin 1''} \sin E.$$

* Ex. 3, Art. 69, and Ex. 8, Art. 71.

Another assumption that is often made is that for small angles

$$\sin n'' = n \sin 1'' \quad (2)$$

The error introduced is

$$\text{for } 1', n'' = 60'', \quad \text{error} = + 0.00000 00000 04;$$

$$\text{for } 1^\circ, n'' = 3600'', \quad \text{error} = + 0.00000 0\bar{9}.$$

Thus, if $\sin \alpha = 0.4 \sin 2^\circ$, we should have, since α must be small, $\alpha'' \sin 1'' = 0.4 \sin 2^\circ$ or $\alpha'' = \frac{0.4 \sin 2^\circ}{\sin 1''}$.

82. Transform the First Member into the Second (or last) in the following examples :

$$1. \frac{\cos \alpha - \sec \alpha}{\sec \alpha} = 4 \cos^2 \frac{1}{2} \alpha (\cos^2 \frac{1}{2} \alpha - 1).$$

The first member contains the angle α and the second $\frac{1}{2} \alpha$; hence we must change the angle.

$$\begin{aligned} \frac{\cos \alpha - \frac{1}{\cos \alpha}}{\frac{1}{\cos \alpha}} &= \cos^2 \alpha - 1 = (2 \cos^2 \frac{1}{2} \alpha - 1)^2 - 1 \\ &= 4 \cos^4 \frac{1}{2} \alpha - 4 \cos^2 \frac{1}{2} \alpha = 4 \cos^2 \frac{1}{2} \alpha (\cos^2 \frac{1}{2} \alpha - 1). \end{aligned}$$

$$2. \operatorname{cosec} 2 \alpha + \cot 2 \alpha = \cot \alpha.$$

$$3. \frac{\operatorname{cosec} 2 \alpha - \cot 2 \alpha}{\operatorname{cosec} 2 \alpha + \cot 2 \alpha} = \tan^2 \alpha.$$

$$4. \cot \alpha - \tan \alpha = 2 \cot 2 \alpha.$$

We may either reduce the expression as far as possible before changing the angle, or change the angle and then reduce.

$$(a) \frac{\cos \alpha}{\sin \alpha} - \frac{\sin \alpha}{\cos \alpha} = \frac{\cos^2 \alpha - \sin^2 \alpha}{\sin \alpha \cos \alpha} = \frac{\cos 2 \alpha}{\frac{1}{2} \sin 2 \alpha} = 2 \cot 2 \alpha.$$

$$(b) \frac{1 + \cos 2 \alpha}{\sin 2 \alpha} - \frac{1 - \cos 2 \alpha}{\sin 2 \alpha} = \frac{2 \cos 2 \alpha}{\sin 2 \alpha} = 2 \cot 2 \alpha.$$

NOTE. — Avoid radicals if possible.

$$5. \sec \alpha \operatorname{cosec} \alpha = 2 \operatorname{cosec} 2 \alpha.$$

$$8. \cot \frac{1}{2} \theta + \tan \frac{1}{2} \theta = 2 \operatorname{cosec} \theta.$$

$$6. (\sin \frac{1}{2} \theta + \cos \frac{1}{2} \theta)^2 = 1 + \sin \theta.$$

$$9. \sin x - 2 \sin^2 x = \sin x \cos 2 x.$$

$$7. \frac{1 - \tan^2 \frac{1}{2} \phi}{1 + \tan^2 \frac{1}{2} \phi} = \cos \phi.$$

$$10. \frac{1}{2} (\sec \theta + \sec^2 \theta) = \frac{1 + \tan^2 \frac{1}{2} \theta}{(1 - \tan^2 \frac{1}{2} \theta)^2}.$$

11. $\frac{2 \tan \frac{1}{2} v}{1 + \tan^2 \frac{1}{2} v} = \sin v.$
12. $\frac{\sec^2 \theta}{2 - \sec^2 \theta} = \sec 2 \theta.$
13. $1 + \tan x \tan \frac{1}{2} x = \sec x.$
14. $\frac{1}{2} (1 + \tan \frac{1}{2} \alpha)^2 = \frac{1 + \sin \alpha}{1 + \cos \alpha}$
15. $\tan \frac{1}{2} \alpha + 2 \sin^2 \frac{1}{2} \alpha \cot \alpha = \sin \alpha.$
16. $\frac{\sin x (1 - \tan^2 x)}{\sec^2 x} \left(\frac{1}{\cos x - \sin x} + \frac{1}{\cos x + \sin x} \right) = \sin 2 x.$
17. $(1 - \tan^2 \theta) \sin \theta \cos \theta = \cos 2 \theta \sqrt{\frac{1 - \cos 2 \theta}{1 + \cos 2 \theta}}$
18. $\frac{1 + \tan^2 \alpha}{1 - \tan^2 \alpha} = \sec 2 \alpha.$
19. $\frac{\cos \theta - \sin \theta}{\cos \theta + \sin \theta} = \frac{1 - \sin 2 \theta}{\cos 2 \theta} = \frac{\cos 2 \theta}{1 + \sin 2 \theta}.$
20. $\frac{\sec \theta + \cos \theta + 2}{\sec \theta + \cos \theta - 2} = \cot^4 \frac{1}{2} \theta.$
- 21.* $\tan \theta \frac{1 + \tan \frac{1}{2} \theta}{1 - \tan \frac{1}{2} \theta} = \frac{\sin \theta}{1 - \sin \theta}.$
22. $\sec 2 \alpha + \tan 2 \alpha + 1 = \frac{2}{1 - \tan \alpha}$
23. $(\sqrt{1 + \sin \alpha} - \sqrt{1 - \sin \alpha})^2 = 4 \sin^2 \frac{1}{2} \alpha.$
24. $(\sqrt{1 + \sin \alpha} + \sqrt{1 - \sin \alpha})^2 = 4 \cos^2 \frac{1}{2} \alpha.$
25. $2 \sin A - \sin (A - B) - 4 \sin A \sin^2 \frac{1}{2} B = \sin (A + B).$
- 26.† $\cos (36^\circ + A) + \cos (36^\circ - A) - \cos (72^\circ + A) - \cos (72^\circ - A) = \cos A.$
- 27.† $\sin (36^\circ + A) - \sin (36^\circ - A) - \sin (72^\circ + A) + \sin (72^\circ - A) = \sin A.$
28. $\frac{\sin x + \sin 2 x}{\cos x - \cos 2 x} = \cot \frac{1}{2} x.$
29. $1 + \cot^2 \frac{1}{2} v = \frac{2}{\sin v \tan \frac{1}{2} v}.$
30. $\tan^3 \frac{1}{2} v (1 + \cot^2 \frac{1}{2} v)^3 = \frac{8}{\sin^3 v}.$
31. $\frac{\sin \alpha \cos \frac{1}{2} \alpha - 2 \cos \alpha \sin \frac{1}{2} \alpha}{2 \sin \frac{1}{2} \alpha - \sin \alpha} = 2 \cos^2 \frac{1}{2} \alpha.$
32. $\frac{\tan^2 \frac{1}{2} x + \cot^2 \frac{1}{2} x}{\tan^2 \frac{1}{2} x - \cot^2 \frac{1}{2} x} = -\frac{1 + \cos^2 x}{2 \cos x}.$
33. Given $\tan \frac{1}{2} v = \sqrt{\frac{1+e}{1-e}} \tan \frac{1}{2} E$, show that
- $$\frac{1}{(1+e) \cos^2 \frac{1}{2} v + (1-e) \sin^2 \frac{1}{2} v} = \frac{1 - e \cos E}{1 - e^2}.$$
34. $\tan (45^\circ + A) - \tan (45^\circ - A) = 2 \tan 2 A.$
- (a) $\frac{\tan 45^\circ + \tan A}{1 - \tan 45^\circ \tan A} - \frac{\tan 45^\circ - \tan A}{1 + \tan 45^\circ \tan A}$
- $$= \frac{1 + \tan A}{1 - \tan A} - \frac{1 - \tan A}{1 + \tan A} = \frac{4 \tan A}{1 - \tan^2 A} = 2 \tan 2 A.$$

* After substituting, multiply both numerator and denominator by the quantity $\sin \theta - 1 + \cos \theta.$

† $\cos 36^\circ = \frac{1}{2}(1 + \sqrt{5}).$

$$(b) \frac{1 - \cos(90^\circ + 2A)}{\sin(90^\circ + 2A)} - \frac{1 - \cos(90^\circ - 2A)}{\sin(90^\circ - 2A)} \\ = \frac{1 + \sin 2A}{\cos 2A} - \frac{1 - \sin 2A}{\cos 2A} = \frac{2 \sin 2A}{\cos 2A} = 2 \tan 2A.$$

$$35. \frac{\tan(45^\circ + \frac{1}{2}A) + \tan(45^\circ - \frac{1}{2}A)}{\tan(45^\circ + \frac{1}{2}A) - \tan(45^\circ - \frac{1}{2}A)} = \operatorname{cosec} A.$$

$$36. \tan(45^\circ + \theta) - \cot(45^\circ + \theta) = 2 \tan 2\theta.$$

$$37. \tan^2(45^\circ + \theta) + \cot^2(45^\circ + \theta) = 2 + 4 \tan^2 2\theta.$$

$$38. \tan^2(45^\circ + \alpha) - \cot^2(45^\circ + \alpha) = 4 \tan 2\alpha \sec 2\alpha.$$

$$39. \frac{\tan(45^\circ + \frac{1}{2}\theta)}{\tan(45^\circ - \frac{1}{2}\theta)} = \frac{1 + \sin \theta}{1 - \sin \theta}.$$

$$40. \tan \theta \tan(45^\circ + \frac{1}{2}\theta) = \frac{\sin \theta}{1 - \sin \theta}.$$

$$41. \cot(45^\circ - \frac{1}{2}\alpha) - \tan(45^\circ - \frac{1}{2}\alpha) = 2 \tan \alpha.$$

$$42. \tan^2(45^\circ + \frac{1}{2}\alpha) = \frac{1 + \sin \alpha}{1 - \sin \alpha}.$$

$$43. \tan(45^\circ + \theta) + \tan(45^\circ - \theta) = 2 \sec 2\theta.$$

$$44. \frac{1 - \tan^2(45^\circ - \theta)}{1 + \tan^2(45^\circ - \theta)} = \sin 2\theta.$$

$$45. \tan(45^\circ + \frac{1}{2}x) \frac{1 + \tan \frac{1}{2}x}{1 - \tan \frac{1}{2}x} = \frac{1 + \sin x}{1 - \sin x}.$$

$$46. \frac{\tan(45^\circ + \frac{1}{2}x)}{1 + \cot^2(45^\circ + \frac{1}{2}x)} = \frac{1}{2} \cos x \frac{1 + \sin x}{1 - \sin x}.$$

$$47. \sin(45^\circ - \frac{1}{2}\theta) + \cos(45^\circ - \frac{1}{2}\theta) = \sqrt{2} \cos \frac{1}{2}\theta = \frac{\sin \theta}{\sqrt{1 - \cos \theta}}.$$

CHAPTER VI.

TRIGONOMETRIC EQUATIONS.

83. One Equation Containing Multiple Angles.* — Change the equation so that it shall contain a single angle, and then proceed as in Art. 52.

1. $\cos 3x = \sin 2x$; find x . (See Ex. 8, Art. 71.)

$$4 \cos^3 x - 3 \cos x = 2 \sin x \cos x.$$

$$\therefore \cos x (1 - 4 \sin^2 x - 2 \sin x) = 0.$$

$$\therefore \cos x = 0, \text{ giving } x = 90^\circ \text{ and } 270^\circ;$$

and $1 - 4 \sin^2 x - 2 \sin x = 0$, giving $\sin x = \frac{1}{4}(\sqrt{5} - 1)$

and $\sin x = -\frac{1}{4}(\sqrt{5} + 1)$, or $x = 18^\circ, 162^\circ, 234^\circ, 306^\circ$.

2. $\cos 2\theta + \cos \theta = -1$; find θ .

Ans. $90^\circ, 270^\circ, 120^\circ, 240^\circ$.

3. $\cot 2\theta + \tan \theta = -\frac{1}{3}\sqrt{3}$; find θ .

Ans. $150^\circ, 330^\circ, 120^\circ, 300^\circ$.

4. $\cos 2x + \sin x = +1$; find x .

Ans. $0^\circ, 30^\circ, 150^\circ, 180^\circ$.

5. $\sin 3x + \sin 2x = \sin x$; find x .

Ans. $0^\circ, 180^\circ, 60^\circ, 300^\circ$.

6. $\tan 2x = -2 \sin x$; find x .

Ans. $0^\circ, 60^\circ, 180^\circ, 300^\circ$.

7. $\tan 2x \tan x = +1$; find x .

Ans. $30^\circ, 150^\circ, 210^\circ, 330^\circ$.

8. $\tan^2 x \tan 2x + 2 \tan x = +\sqrt{3}$; find x .

Ans. $30^\circ, 120^\circ, 210^\circ, 300^\circ$.

9. $\sin 4x - 2 \sin 2x = 0$; find x .

Ans. $0^\circ, 90^\circ, 180^\circ, 270^\circ$.

The equation may sometimes be solved by the use of the equations of Art. 73.

10. $\cos 3x - \sin 2x = 0$; find x .

$$\cos 3x - \sin 2x = \sin (90^\circ + 3x) - \sin 2x$$

$$= 2 \cos (45^\circ + \frac{1}{2}x) \sin (45^\circ + \frac{1}{2}x) = 0.$$

$\cos (45^\circ + \frac{1}{2}x) = 0$ gives $45^\circ + \frac{1}{2}x = 90^\circ, 270^\circ, 450^\circ, 630^\circ, 810^\circ$,
or $x = 18^\circ, 90^\circ, 162^\circ, 234^\circ, 306^\circ$.

$\sin (45^\circ + \frac{1}{2}x) = 0$ gives $45^\circ + \frac{1}{2}x = 0^\circ, 180^\circ$,

or $x = -90^\circ$ and 270° .

* See Art. 52 for the solution of equations when only one angle is involved

11. $\cos \theta - \cos 3\theta = \sin 2\theta$; find θ by both methods.
Ans. $0^\circ, 30^\circ, 90^\circ, 150^\circ, 180^\circ, 270^\circ$.
12. $\sin 3\theta + \sin 2\theta + \sin \theta = 0$; find θ by both methods.
Ans. $0^\circ, 90^\circ, 120^\circ, 180^\circ, 240^\circ, 270^\circ$.
13. $\cos 2\theta = \sin \theta$; find θ by both methods. *Ans.* $30^\circ, 150^\circ, 270^\circ$.
14. $\cos 5\theta - \cos 3\theta + \sin \theta = 0$; find θ . *Ans.* $0^\circ, 180^\circ, (2n + \frac{1}{2} \pm \frac{1}{3})\frac{\pi}{4}$.
15. $\sin 5\theta + \sin 3\theta + 2\cos \theta = 0$; find θ . *Ans.* $90^\circ, 270^\circ, (2n + \frac{1}{2})\frac{\pi}{4}$.
16. $\sin(60^\circ - x) - \sin(60^\circ + x) = +\frac{1}{2}\sqrt{3}$; find x . *Ans.* $240^\circ, 300^\circ$.
17. $\sin(30^\circ + x) - \cos(60^\circ + x) = -\frac{1}{2}\sqrt{3}$; find x . *Ans.* $210^\circ, 330^\circ$.
18. $\cos 4z - \cos 2z = 0$; find z . *Ans.* $0^\circ, 60^\circ, 120^\circ, 180^\circ, 240^\circ, 300^\circ$.

84. Find r and ϕ from the Equations

$$r \sin \phi = a, \quad (1)$$

$$r \cos \phi = b, \quad (2)$$

a and b being known.

$$(1) \div (2) \text{ gives } \tan \phi = \frac{a}{b}. \quad (3)$$

$$\text{From (1) and (2)} \quad r = \frac{a}{\sin \phi} = \frac{b}{\cos \phi}. \quad (4)$$

1. Find r and ϕ when $\log a = 0.47141$, and $\log b = 0.63927$ n , r being positive.

$$\log(r \sin \phi) = \log a = 0.47141 \quad (1)$$

$$\log \sin \phi = 9.74972 \quad (5)$$

$$\log \cos \phi = 9.91758 \quad (6)$$

$$\log(r \cos \phi) = \log b = 0.63927 \quad (2)$$

$$(1) - (2) = \log \tan \phi = 9.83214 \quad (3)$$

$$\phi = 145^\circ 48'.4 \quad (4)$$

$$(1) - (5) = (2) - (6) = \log r = 0.72169 \quad (7)$$

$$r = 5.2685 \quad (8)$$

The numbers on the right indicate the order in which the quantities are found. If the two values of $\log r$ had differed, we should have taken that found from $\log \cos \phi$, as a small error in $\log \tan \phi$ would, for this value of ϕ , affect the logarithmic cosine less than the logarithmic sine. The angle ϕ is placed in the second quadrant, since $r \cos \phi$ is negative and $r \sin \phi$ positive, r being considered positive.

2. Find r and ϕ when $\log a = 0.46843$ n , and $\log b = 0.43742$, r being positive.

$$\text{Ans. } \phi = 312^\circ 57'.4; r = 4.0178.$$

3. Find r and ϕ when $\log a = 1.46444$ n , and $\log b = 1.86903$ n , r being positive.

$$\text{Ans. } \phi = 201^\circ 30'.0; r = 79.497.$$

85. Find r , ϕ , and θ from the Equations

$$\left. \begin{aligned} r \cos \phi \cos \theta &= a, \\ r \sin \phi \cos \theta &= b, \\ r \sin \theta &= c, \end{aligned} \right\} \quad (1)$$

$$(2)$$

$$(3)$$

a , b , and c being known.

$$(2) \div (1) \text{ gives } \tan \phi = \frac{b}{a}. \quad (4)$$

$$\text{From (1) and (2), } r \cos \theta = \frac{a}{\cos \phi} = \frac{b}{\sin \phi} \quad (5)$$

$$\text{From (3), } r \sin \theta = c. \quad (6)$$

$$(6) \div (5) \text{ gives } \tan \theta = \frac{c \cos \phi}{a} = \frac{c \sin \phi}{b}. \quad (7)$$

From (5) and (6),

$$r = \frac{a}{\cos \phi \cos \theta} = \frac{b}{\sin \phi \cos \theta} = \frac{c}{\sin \theta}. \quad (8)$$

1. Given $\log a = 0.46472$, $\log b = 0.72413 n$, $\log c = 0.62817$, find r , ϕ , and θ , θ being numerically less than 90° , and r being positive.

$$\log (r \cos \phi \cos \theta) = \log a = 0.46472 \quad (1)$$

$$\log \cos \phi = (9.68314) \quad \text{Only as a check.} \quad (5)$$

$$\log \sin \phi = 9.94256 n \quad (6)$$

$$\log (r \sin \phi \cos \theta) = \log b = 0.72413 n \quad (2)$$

$$(2) - (1) = \log \tan \phi = 0.25941 n \quad (3)$$

$$\phi = 298^\circ 49'.4 \quad (4)$$

$$(2) - (5) = (1) - (5) = \log (r \cos \theta) = 0.78157 \quad (6)$$

$$\log \cos \theta = 9.91291 \quad (10)$$

$$\log \sin \theta = (9.75951) \quad \text{Only as a check.} \quad (10)$$

$$\log c = \log (r \sin \theta) = 0.62817 \quad (7)$$

$$(7) - (6) = \log \tan \theta = 9.84660 \quad (8)$$

$$\theta = 35^\circ 5'.1 \quad (9)$$

$$(6) - (10) = (7) - (10) = \log r = 0.86866 \quad (11)$$

$$r = 7.3903 \quad (12)$$

The angle ϕ is placed in the fourth quadrant, since $r \cos \theta$ is positive, and therefore $\cos \phi$ must be positive and $\sin \phi$ negative, $r \cos \phi \cos \theta$ being positive and $r \sin \phi \cos \theta$ negative.

2. Given $\log a = 0.26903 n$, $\log b = 0.32426$, $\log c = 0.36903 n$, find r , ϕ , and θ , r being positive and θ numerically less than 90° .

$$\text{Ans. } \phi = 131^\circ 22'.0; \theta = -39^\circ 45'.6; r = 3.6572.$$

3. Given $\log a = 9.43942 n$, $\log b = 9.40403 n$, $\log c = 9.56700 n$, find r , ϕ , and θ , r being positive and θ numerically less than 90° .

$$\text{Ans. } \phi = 222^\circ 40'.1; \theta = -44^\circ 36'.4; r = 0.525425 \text{ or } 0.52544.$$

86. Find ϕ from the Equation

$$a \sin \phi + b \cos \phi = c \quad (1)$$

by formulas adapted to logarithmic computation, a , b , and c being known.

Let M be an auxiliary angle and m a positive constant, so that

$$\left. \begin{aligned} m \sin M &= a, \\ m \cos M &= b. \end{aligned} \right\} \quad (2)$$

The angle M is always possible, for we have, by division,

$$\tan M = \frac{a}{b}, \quad (3)$$

and since the tangent may have any value between $+\infty$ and $-\infty$, there will always be some angle whose tangent is equal to $\frac{a}{b}$. Also, squaring and adding Eqs. (2), we have

$$m^2 \sin^2 M + m^2 \cos^2 M = m^2 = a^2 + b^2,$$

$$\text{or} \quad m = \sqrt{a^2 + b^2}. \quad (4)$$

Therefore the assumptions in (2) are always possible, since M and m will be real quantities if a and b are real.

Substituting (2) in (1), we have

$$m \sin M \sin \phi + m \cos M \cos \phi = c,$$

$$\text{or} \quad m \cos(\phi - M) = c. \quad (5)$$

Hence, from (2) find M and m by the method of Art. 84; from (5) find $\phi - M$ (two values $< 360^\circ$), and thence find ϕ .

1. Find ϕ when $2 \sin \phi - 3 \cos \phi = 1$.

$$\text{Ans. } M = 146^\circ 18'.6; \phi = 220^\circ 12'.5, \text{ or } 72^\circ 24'.7.$$

2. Find ϕ when $2 \sin \phi + 4 \cos \phi = -3$.

$$\text{Ans. } M = 26^\circ 33'.9; \phi = 158^\circ 41'.8, \text{ or } 254^\circ 26'.0.$$

 87. Find ϕ from the Equation

$$a \tan \phi + b \cot \phi = c$$

by formulas adapted to logarithmic computation, a , b , and c being known.

Substituting for $\tan \phi$ and $\cot \phi$ in terms of $\sin \phi$ and $\cos \phi$, we have, after reducing,

$$(a - b) \cos 2\phi + c \sin 2\phi = a + b.$$

$$\left. \begin{array}{l} \text{Let } m \sin M = a - b, \\ m \cos M = c. \end{array} \right\} \therefore m \sin (M + 2\phi) = a + b.$$

1. Find ϕ when $2 \tan \phi - \cot \phi = -3$.

Ans. $M = 135^\circ$; $\phi = 15^\circ 41'.0$, $119^\circ 19'.0$, $195^\circ 41'.0$, $299^\circ 19'.0$.

2. Find ϕ when $\tan \phi + 3 \cot \phi = -2\sqrt{3}$.

Ans. $M = 210^\circ$; $\phi = 120^\circ$ or 300° .

88. Find ϕ from the Following Equations, a and α being known:

$$(a) \quad \sin(\phi + \alpha) = a \sin \phi. \quad (1)$$

Expanding, $\sin \phi \cos \alpha + \cos \phi \sin \alpha = a \sin \phi$.

$$\therefore \sin \phi (a - \cos \alpha) = \cos \phi \sin \alpha.$$

$$\therefore \tan \phi = \frac{\sin \alpha}{a - \cos \alpha}. \quad (2)$$

Eq. (2) is not adapted to logarithmic computation. But from (1) we have

$$\frac{\sin(\phi + \alpha)}{\sin \phi} = \frac{a}{1},$$

and, by composition and division,

$$\frac{\sin(\phi + \alpha) + \sin \phi}{\sin(\phi + \alpha) - \sin \phi} = \frac{a + 1}{a - 1},$$

and this, from the equations of Art. 73, becomes

$$\frac{\tan(\phi + \frac{1}{2}\alpha)}{\tan \frac{1}{2}\alpha} = \frac{a + 1}{a - 1},$$

$$\text{or} \quad \tan(\phi + \frac{1}{2}\alpha) = \frac{a + 1}{a - 1} \tan \frac{1}{2}\alpha. \quad (3)$$

Let $\tan \beta = a$, and note that $\tan 45^\circ = 1$.

$$\begin{aligned} \therefore \tan(\phi + \frac{1}{2}\alpha) &= \frac{\tan \beta + \tan 45^\circ}{\tan \beta - \tan 45^\circ} \tan \frac{1}{2}\alpha \\ &= \frac{\sin(\beta + 45^\circ)}{\sin(\beta - 45^\circ)} \tan \frac{1}{2}\alpha. \end{aligned}$$

$$\therefore \tan(\phi + \frac{1}{2}\alpha) = \cot(\beta - 45^\circ) \tan \frac{1}{2}\alpha. \quad (4)$$

$$(b) \quad \cos(\phi + \alpha) = a \cos \phi.$$

$$\therefore \tan(\phi + \frac{1}{2}\alpha) = \tan(45^\circ - \beta) \cot \frac{1}{2}\alpha, \text{ if } \tan \beta = a.$$

$$(c) \quad \sin(\alpha - \phi) = a \sin \phi.$$

$$\therefore \tan(\phi - \frac{1}{2}\alpha) = \tan(45^\circ - \beta) \tan \frac{1}{2}\alpha, \text{ if } \tan \beta = a.$$

$$(d) \quad \sin(\phi + \alpha) = a \cos \phi.$$

$$\therefore \sin(\phi + \alpha) = a \sin(90^\circ + \phi);$$

$$\therefore \tan(45^\circ + \phi + \frac{1}{2}\alpha) = \cot(45^\circ - \beta) \tan(45^\circ - \frac{1}{2}\alpha), \text{ if } \tan \beta = a.$$

$$(e) \quad \cos(\phi + \alpha) = a \sin \phi.$$

$$\therefore \cos(\phi + \alpha) = a \cos(90^\circ - \phi);$$

$$\therefore \tan(\phi + \frac{1}{2}\alpha - 45^\circ) = \tan(45^\circ - \beta) \cot(45^\circ + \frac{1}{2}\alpha), \text{ if } \tan \beta = a.$$

NOTE. — The equation $a \sin(\phi + \alpha) = a' \sin(\phi + \alpha')$ and similar equations may be solved by expansion, the solution of the given equation being

$$\tan \phi = \frac{a' \sin \alpha' - a \sin \alpha}{a \cos \alpha - a' \cos \alpha'}.$$

A solution adapted to logarithmic computation may be found by the method of this article, giving

$$\tan[\phi + \frac{1}{2}(\alpha + \alpha')] = \cot(\beta - 45^\circ) \tan \frac{1}{2}(\alpha - \alpha'), \text{ if } \tan \beta = \frac{a'}{a}.$$

89. Find ϕ from the Following Equations, a and α being known:

$$(a) \quad \sin(\phi + \alpha) \sin \phi = a.$$

From (8), Art. 72,

$$\cos \alpha - \cos(2\phi + \alpha) = 2a.$$

$$\therefore \cos(2\phi + \alpha) = \cos \alpha - 2a. \quad (1)$$

$$\text{Let} \quad \tan \beta = \frac{2a}{\sin \alpha}. \quad (2)$$

$$\therefore \cos(2\phi + \alpha) = \cos \alpha - \sin \alpha \tan \beta = \frac{\cos \alpha \cos \beta - \sin \alpha \sin \beta}{\cos \beta}.$$

$$\therefore \cos(2\phi + \alpha) = \frac{\cos(\alpha + \beta)}{\cos \beta}. \quad (3)$$

$$(b) \quad \sin(\alpha - \phi) \sin \phi = a.$$

$$\therefore \cos(\alpha - 2\phi) - \cos \alpha = 2a;$$

$$\therefore \cos(\alpha - 2\phi) = \cos \alpha + 2a.$$

$$\therefore \cos(\alpha - 2\phi) = \frac{\cos(\alpha - \beta)}{\cos \beta}, \text{ if } \tan \beta = \frac{2a}{\sin \alpha}.$$

$$(c) \sin(\phi + \alpha) \cos \phi = a.$$

$$\therefore \sin(2\phi + \alpha) + \sin \alpha = 2a.$$

$$\therefore \sin(2\phi + \alpha) = 2a - \sin \alpha = \frac{\sin(\beta - \alpha)}{\cos \beta}, \text{ if } \tan \beta = \frac{2a}{\cos \alpha}.$$

$$(d) \cos(\phi + \alpha) \cos \phi = a.$$

$$\therefore \cos(2\phi + \alpha) + \cos \alpha = 2a.$$

$$\therefore \cos(2\phi + \alpha) = 2a - \cos \alpha = -\frac{\cos(\alpha + \beta)}{\cos \beta}, \text{ if } \tan \beta = \frac{2a}{\sin \alpha}.$$

$$(e) \cos(\phi + \alpha) \sin \phi = a.$$

$$\therefore \sin(2\phi + \alpha) = 2a + \sin \alpha = \frac{\sin(\alpha + \beta)}{\cos \beta}, \text{ if } \tan \beta = \frac{2a}{\cos \alpha}.$$

90. Find ϕ from the Following Equations, a , α , and α' being known :

$$(a) \tan(\phi + \alpha) = a \tan \phi.$$

$$\therefore \frac{\tan(\phi + \alpha)}{\tan \phi} = \frac{a}{1}; \therefore \frac{\tan(\phi + \alpha) + \tan \phi}{\tan(\phi + \alpha) - \tan \phi} = \frac{a + 1}{a - 1};$$

$$\therefore \frac{\sin(2\phi + \alpha)}{\sin \alpha} = \frac{a + 1}{a - 1}. \quad (1)$$

Let

$$\tan \beta = a; \quad (2)$$

$$\therefore \frac{a + 1}{a - 1} = \cot(\beta - 45^\circ),$$

and

$$\sin(2\phi + \alpha) = \cot(\beta - 45^\circ) \sin \alpha. \quad (3)$$

Find β from (2) and $2\phi + \alpha$ from (3).

$$(b) \tan(\phi + \alpha) = a \cot \phi.$$

$$\therefore \cos(2\phi + \alpha) = \tan(45^\circ - \beta) \cos \alpha, \text{ if } \tan \beta = a.$$

$$(c) \cot(\alpha - \phi) = a \cot \phi.$$

$$\therefore \sin(2\phi - \alpha) = \tan(\beta - 45^\circ) \sin \alpha, \text{ if } \tan \beta = a.$$

$$(d) \cot(\phi + \alpha) = a \cot(\phi - \alpha).$$

$$\therefore \sin 2\phi = \cot(45^\circ - \beta) \sin 2\alpha, \text{ if } \tan \beta = a.$$

$$(e) \tan(\phi + \alpha) = a \tan(\phi + \alpha').$$

$$\therefore \sin(2\phi + \alpha + \alpha') = \cot(\beta - 45^\circ) \sin(\alpha - \alpha'), \text{ if } \tan \beta = a.$$

$$(f) \cot(\phi + \alpha) = a \cot(\phi + \alpha').$$

$$\therefore \sin(2\phi + \alpha + \alpha') = \cot(\beta - 45^\circ) \sin(\alpha' - \alpha), \text{ if } \tan \beta = a.$$

$$(g) \cot(\phi + \alpha) = a \tan(\phi + \alpha').$$

$$\therefore \cos(2\phi + \alpha + \alpha') = \tan(\beta - 45^\circ) \cos(\alpha - \alpha'), \text{ if } \tan \beta = a.$$

91. Find ϕ from the Following Equations, a , α , and α' being known: *

$$(a) \tan(\phi + \alpha) \tan \phi = a.$$

$$\therefore \sin(\phi + \alpha) \sin \phi = a \cos(\phi + \alpha) \cos \phi.$$

From the equations of Art. 72, we have

$$-\cos(2\phi + \alpha) + \cos \alpha = a \cos(2\phi + \alpha) + a \cos \alpha;$$

$$\therefore \cos(2\phi + \alpha) = \frac{1-a}{1+a} \cos \alpha.$$

Let

$$\tan \beta = a.$$

$$\therefore \cos(2\phi + \alpha) = \tan(45^\circ - \beta) \cos \alpha.$$

$$(b) \tan(\phi + \alpha) \cot \phi = a.$$

$$\therefore \sin(2\phi + \alpha) = \cot(\beta - 45^\circ) \sin \alpha, \text{ if } \tan \beta = a.$$

$$(c) \tan(\phi + \alpha) \tan(\phi - \alpha) = a.$$

$$\therefore \cos 2\phi = \tan(45^\circ - \beta) \cos 2\alpha, \text{ if } \tan \beta = a.$$

$$(d) \tan(\phi + \alpha) \cot(\phi + \alpha') = a.$$

$$\therefore \sin(2\phi + \alpha + \alpha') = \cot(\beta - 45^\circ) \sin(\alpha - \alpha'), \text{ if } \tan \beta = a.$$

92. Find r and ϕ from the Following Equations, a , b , α , and β being known :

$$r \sin(\phi + \alpha) = a, \quad (1)$$

$$r \cos(\phi + \beta) = b. \quad (2)$$

$$\therefore \frac{\sin(\phi + \alpha)}{\cos(\phi + \beta)} = \frac{a}{b}; \quad \therefore b \sin(\phi + \alpha) = a \cos(\phi + \beta).$$

$$\therefore b \sin \phi \cos \alpha + b \cos \phi \sin \alpha = a \cos \phi \cos \beta - a \sin \phi \sin \beta.$$

$$\therefore b \sin \phi \cos \alpha + a \sin \phi \sin \beta = a \cos \phi \cos \beta - b \cos \phi \sin \alpha.$$

$$\therefore \sin \phi (b \cos \alpha + a \sin \beta) = \cos \phi (a \cos \beta - b \sin \alpha).$$

$$\therefore \tan \phi = \frac{a \cos \beta - b \sin \alpha}{b \cos \alpha + a \sin \beta}, \quad (3)$$

and

$$r = \frac{a}{\sin(\phi + \alpha)} = \frac{b}{\cos(\phi + \beta)}. \quad (4)$$

The quadrant of ϕ will be determined by the sign assigned to r .

* The method of Art. 90 may be used, since $\tan x = \frac{1}{\cot x}$ and $\cot x = \frac{1}{\tan x}$.

1. If $r \sin(\phi + \alpha) = a$, and $r \sin(\phi + \beta) = b$, show that

$$\tan \phi = \frac{a \sin \beta - b \sin \alpha}{b \cos \alpha - a \cos \beta}$$

2. If $r \cos(\phi + \alpha) = a$, and $r \cos(\phi + \beta) = b$, show that

$$\tan \phi = \frac{a \cos \beta - b \cos \alpha}{a \sin \beta - b \sin \alpha}$$

93. Find r and ϕ from the Following Equations, a, b, α , and β being known, and the formulas derived being adapted to logarithmic computation :

$$r \sin(\phi + \alpha) = a, \quad (1)$$

$$r \sin(\phi + \beta) = b. \quad (2)$$

$$(1) + (2) = r [\sin(\phi + \alpha) + \sin(\phi + \beta)] = a + b;$$

$$\therefore 2r \sin \left[\phi + \frac{1}{2}(\alpha + \beta) \right] \cos \frac{1}{2}(\alpha - \beta) = a + b. \quad (3)$$

$$(1) - (2) = r [\sin(\phi + \alpha) - \sin(\phi + \beta)] = a - b;$$

$$\therefore 2r \cos \left[\phi + \frac{1}{2}(\alpha + \beta) \right] \sin \frac{1}{2}(\alpha - \beta) = a - b. \quad (4)$$

From (3) and (4), we have

$$\left. \begin{aligned} r \sin \left[\phi + \frac{1}{2}(\alpha + \beta) \right] &= \frac{a + b}{2 \cos \frac{1}{2}(\alpha - \beta)}, \\ r \cos \left[\phi + \frac{1}{2}(\alpha + \beta) \right] &= \frac{a - b}{2 \sin \frac{1}{2}(\alpha - \beta)}. \end{aligned} \right\}$$

from which r and $\phi + \frac{1}{2}(\alpha + \beta)$ are found by the method of Art. 84.

1. If $r \cos(\phi + \alpha) = a$, and $r \cos(\phi + \beta) = b$, show that

$$r \cos \left[\phi + \frac{1}{2}(\alpha + \beta) \right] = \frac{a + b}{2 \cos \frac{1}{2}(\alpha - \beta)},$$

$$r \sin \left[\phi + \frac{1}{2}(\alpha + \beta) \right] = \frac{b - a}{2 \sin \frac{1}{2}(\alpha - \beta)}.$$

2. If $r \sin(\phi + \alpha) = a$, and $r \cos(\phi + \beta) = b$, show that by placing $\cos(\phi + \beta) = \sin(90^\circ + \phi + \beta)$ we may obtain

$$r \sin \left[\phi + 45^\circ + \frac{1}{2}(\alpha + \beta) \right] = \frac{a + b}{2 \cos \left[45^\circ - \frac{1}{2}(\alpha - \beta) \right]},$$

$$r \cos \left[\phi + 45^\circ + \frac{1}{2}(\alpha + \beta) \right] = \frac{b - a}{2 \sin \left[45^\circ - \frac{1}{2}(\alpha - \beta) \right]}.$$

3. Find r and ϕ when $r \sin(\phi + 100^\circ) = 2$, and $r \sin(\phi + 200^\circ) = 3$, r being positive.

Ans. $\phi = 290^\circ 28'.4$; $r = 3.9436$.

CHAPTER VII.

OBLIQUE PLANE TRIANGLES.

94. It has been shown in Geometry that a triangle can be constructed when three elements, one being a side, are known. If the three angles only are given, there will be an infinite number of triangles satisfying the conditions of the problem, since the data determine the *shape* and not the size of the triangle.

We also know that in any triangle

- (1) The sum of the three angles is 180° .
- (2) If one angle is 90° , the sum of the other two is 90° .
- (3) The greater side is opposite the greater angle, and conversely.
- (4) Any side is less than the sum of the other two.

95. The Sine Proportion. — *The sides of a triangle are to each other as the sines of the opposite angles.*

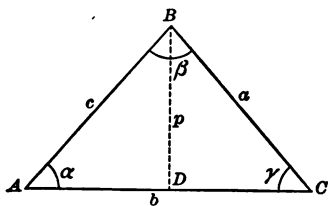


FIG. 52.

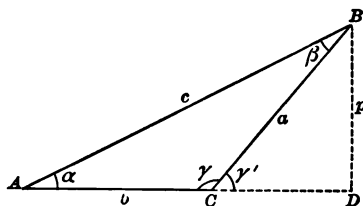


FIG. 53.

In Fig. 52, $p = a \sin \gamma$; $p = c \sin \alpha$.

$$\therefore a \sin \gamma = c \sin \alpha. \quad (1)$$

$$\therefore \frac{a}{c} = \frac{\sin \alpha}{\sin \gamma}, \text{ or } \frac{a}{\sin \alpha} = \frac{c}{\sin \gamma} \quad (2)$$

In Fig. 53,

$$p = a \sin \gamma' = a \sin (180^\circ - \gamma) = a \sin \gamma, \text{ and } p = c \sin \alpha.$$

$$\therefore a \sin \gamma = c \sin \alpha, \text{ as before.}$$

In the same way, by drawing a line perpendicular to AB from C (Figs. 52 and 53), we can show that

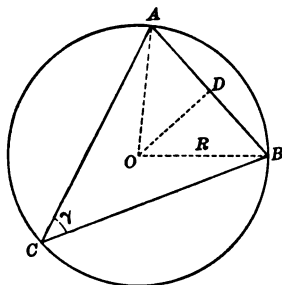


FIG. 54.

$$\frac{a}{\sin \alpha} = \frac{b}{\sin \beta}.$$

$$\therefore \frac{a}{\sin \alpha} = \frac{b}{\sin \beta} = \frac{c}{\sin \gamma}, \quad (3)$$

true for both acute and obtuse angled triangles.

NOTE. — The constant quotient $\frac{a}{\sin \alpha}$ is called the *modulus* of the triangle, and is equal to the diameter of the circumscribed circle.

For, in Fig. 54, $c = AB = 2R \sin \angle AOD = 2R \sin \frac{1}{2} \angle AOB = 2R \sin \gamma$.

$$\therefore \frac{c}{\sin \gamma} = 2R.$$

96. The Square of Any Side of a Triangle is equal to the sum of the squares of the other two sides, diminished by twice the product of the two sides multiplied by the cosine of their included angle.

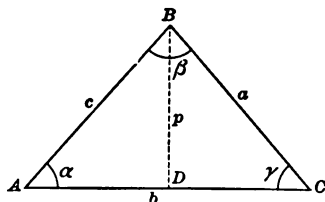


FIG. 55.

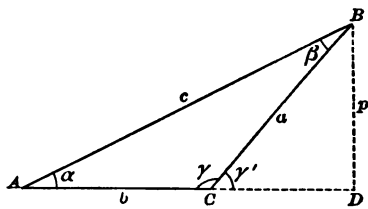


FIG. 56.

From geometry we have, in Fig. 55,

$$c^2 = a^2 + b^2 - 2b \cdot DC = a^2 + b^2 - 2ab \cos \gamma. \quad (1)$$

Also, in Fig. 56,

$$c^2 = a^2 + b^2 + 2b \cdot CD = a^2 + b^2 + 2ab \cos \gamma',$$

or

$$c^2 = a^2 + b^2 - 2ab \cos \gamma. \quad (2)$$

This relation may also be proved as follows :

In Fig. 55, $b = AC = AD + DC = c \cos \alpha + a \cos \gamma.$

In Fig. 56, $b = AC = AD - CD = c \cos \alpha - a \cos \gamma'$
 $= c \cos \alpha + a \cos \gamma.$

$$\therefore b = c \cos \alpha + a \cos \gamma.$$

$$\therefore c \cos \alpha = b - a \cos \gamma.$$

$$\therefore c^2 \cos^2 \alpha = a^2 \cos^2 \gamma + b^2 - 2ab \cos \gamma.$$

But $c^2 \sin^2 \alpha = a^2 \sin^2 \gamma,$ from (1), Art. 95.

By addition, $c^2 = a^2 + b^2 - 2ab \cos \gamma,$

since $\sin^2 x + \cos^2 x = 1.$

97. Case I. Given One Side and Two Angles (α, a, β).

Formulas: $\gamma = 180^\circ - (\alpha + \beta);$

$$b = \frac{a}{\sin \alpha} \sin \beta;$$

$$c = \frac{a}{\sin \alpha} \sin \gamma.$$

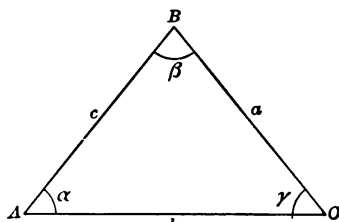


FIG. 57.

1. Solve the triangle when $a = 3.4356,$
 $\alpha = 17^\circ 43'.4, \gamma = 60^\circ 35'.7.$

$$\therefore \beta = 180^\circ - (\alpha + \gamma) = 101^\circ 40'.9.$$

(a) By natural functions.

$$b = a \times \sin \beta \div \sin \alpha = 3.4356 \times .97929 \div .30442 = 11.052.$$

$$c = a \times \sin \gamma \div \sin \alpha = 3.4356 \times .87117 \div .30442 = 9.8318.$$

(b) By the use of logarithms.

$$\log b = \log a - \log \sin \alpha + \log \sin \beta = \log a + \text{col } \sin \alpha + \log \sin \beta.$$

$$\log c = \log a - \log \sin \alpha + \log \sin \gamma = \log a + \text{col } \sin \alpha + \log \sin \gamma.$$

$\log a = 0.53600$	$\log a = 0.53600$
$\text{col } \sin \alpha = 0.51652$	$\text{col } \sin \alpha = 0.51652$
$\log \sin \beta = 9.99091$	$\log \sin \gamma = 9.94010$
$\log b = 1.04343$	$\log c = 0.99262$
$b = 11.052$	$c = 9.8315$

2. Solve the triangle when $c = 54.376, \alpha = 103^\circ 3'.2, \beta = 40^\circ 10'.3.$

$$\text{Ans. } \gamma = 36^\circ 48'.5; b = 58.591; a = 88.478.$$

3. Solve the triangle when $a = 0.14323, \alpha = 53^\circ 17'.3, \beta = 62^\circ 23'.5.$

$$\text{Ans. } \gamma = 64^\circ 19'.2; b = 0.15832; c = 0.16101.$$

98. Case II. Given Two Sides and the Angle Opposite One of them (α, c, a). — From the sine proportion, we have

$$\sin \gamma = \frac{c}{a} \sin \alpha. \quad (1)$$

Since γ is found from (1) by means of its sine, it may have two values, one in the first and one in the second quadrant, their sum being 180° . Therefore there *may* be two triangles with the given elements.

If α is obtuse, γ must be acute, since there can be only one obtuse angle in a plane triangle, and there will be only one solution.

If α is acute, and a is greater than c , γ will be acute, since α must be greater than γ , and there will be only one solution.

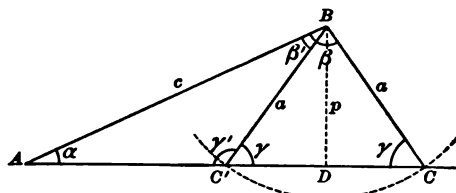


FIG. 58.

If α is acute, and a is equal to c , there will be only one solution, since the points C' and A will coincide.

If α is acute, and a is less than c , γ will be greater than α , and therefore γ may be either in the first or in the second quadrant.

In order that there may be two solutions, the given angle must be acute, and the side opposite it must be less than the side adjacent.

If $a = DB$, the two triangles will be coincident, γ being 90° . If a is less than DB , the triangle will be impossible; this will be shown in the computation where $\sin \gamma$, found from (1), will be greater than unity.

If we use primed letters to represent the unknown elements of one of the triangles, and unprimed letters for those of the other, we have

$$\text{Formulas: } \sin \gamma = \frac{c}{a} \sin \alpha = \sin \gamma';$$

$$\beta = 180^\circ - (\alpha + \gamma); \quad \beta' = 180^\circ - (\alpha + \gamma');$$

$$b = \frac{a}{\sin \alpha} \sin \beta; \quad b' = \frac{a}{\sin \alpha} \sin \beta';$$

$$\text{or} \quad b = \frac{c}{\sin \gamma} \sin \beta; \quad b' = \frac{c}{\sin \gamma'} \sin \beta'.$$

1. Solve the triangle when $a = 9.4672$, $c = 14.433$, $\alpha = 11^\circ 14'.3$.

In this example $\alpha < 90^\circ$, $\alpha < c$; $\therefore$ two solutions.

$$\log \sin \gamma = \log c + \text{col } a + \log \sin \alpha = \log \sin \gamma'.$$

$$\beta = 180^\circ - (\alpha + \gamma); \quad \beta' = 180^\circ - (\alpha + \gamma').$$

$$\log b = \log a + \text{col } \sin \alpha + \log \sin \beta = \log c + \text{col } \sin \gamma + \log \sin \beta.$$

$$\log b' = \log a + \text{col } \sin \alpha + \log \sin \beta' = \log c + \text{col } \sin \gamma' + \log \sin \beta'.$$

$\log c = 1.15936$	$\log a = 0.97622$	$\log c = 1.15936$
$\text{col } a = 9.02378$	$\text{col } \sin \alpha = 0.71021$	$\text{col } \sin \gamma = 0.52707$
$\log \sin \alpha = 9.28979$	$\log \sin \beta = 9.67899$	$\log \sin \beta = 9.67899$
$\log \sin \gamma = 9.47293$	$\log b = 1.36542$	$\log b = 1.36542$
$\gamma = 17^\circ 17'.1$	$b = 23.196$	$b = 23.196$
$\gamma' = 162^\circ 42'.9$		
$\therefore \beta = 151^\circ 28'.6$	$\log a = 0.97622$	$\log c = 1.15936$
$\beta' = 6^\circ 2'.8$	$\text{col } \sin \alpha = 0.71021$	$\text{col } \sin \gamma' = 0.52707$
	$\log \sin \beta' = 9.02259$	$\log \sin \beta' = 9.02259$
	$\log b' = 0.70902$	$\log b' = 0.70902$
	$b' = 5.1170$	$b' = 5.1170$

2. Solve the triangle when $a = 2.4741$, $c = 1.0003$, $\alpha = 69^\circ 14'.8$.

$$\text{Ans. } \gamma = 22^\circ 12'.8; \quad \beta = 88^\circ 32'.4; \quad b = 2.6449.$$

3. Solve the triangle when $a = 10.473$, $b = 12.987$, $\alpha = 44^\circ 11'.3$.

$$\text{Ans. } \begin{cases} \beta = 59^\circ 48'.5; \quad \gamma = 76^\circ 0'.2; \quad c = 14.579; \\ \beta' = 120^\circ 11'.5; \quad \gamma' = 15^\circ 37'.2; \quad c' = 4.0456. \end{cases}$$

4. Solve the triangle when $a = 0.43477$, $b = 0.40031$, $\alpha = 94^\circ 17'.6$.

$$\text{Ans. } \beta = 66^\circ 39'.6; \quad \gamma = 19^\circ 2'.8; \quad c = 0.14228.$$

99. Case III. Given the Three Sides (a, b, c).

(a) From Art. 96,

$$a^2 = b^2 + c^2 - 2bc \cos \alpha.$$

$$\therefore \cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}. \quad (1)$$

From this equation we may find α by means of its *natural* cosine.

$a = 0.0093146$	$\log(s - a) = 7.95219 - 10$
$b = 0.0176530$	$\log(s - b) = 6.79183 - 10$
$c = 0.0095768$	$\log(s - c) = 7.93929 - 10$
$2s = 0.0365444$	$\text{col } s = 1.73821$
$s = 0.0182722$	$\log r^2 = 4.42152 - 10$
$s - a = 0.0089576$	$\log r = 7.21076 - 10$
$s - b = 0.0006192$	$\therefore \log \tan \frac{1}{2} \alpha = 9.25857$
$s - c = 0.0089554$	$\frac{1}{2} \alpha = 10^\circ 16'.8$
$\text{sum} = 0.0365444$	$\log \tan \frac{1}{2} \beta = 0.41893$
$2s = 0.0365444$	$\frac{1}{2} \beta = 69^\circ 8'.2$
$a \text{ check.}$	$\log \tan \frac{1}{2} \gamma = 9.27147$
	$\frac{1}{2} \gamma = 10^\circ 35'.0$

In finding $\log \tan \frac{1}{2} \alpha$, write $\log r$ on the margin of a slip of paper, place it above $\log(s - a)$, and write the difference of the two logarithms opposite $\log \tan \frac{1}{2} \alpha$; then find $\log \tan \frac{1}{2} \beta$ and $\log \tan \frac{1}{2} \gamma$ in the same way. Find $s - a$, $s - b$, and $s - c$ in a similar manner.

2. Solve the triangle when $a = 32.456$, $b = 41.724$, $c = 53.987$.

Ans. $\frac{1}{2} \alpha = 18^\circ 27'.4$; $\frac{1}{2} \beta = 25^\circ 16'.3$; $\frac{1}{2} \gamma = 46^\circ 16'.4$.

3. Solve the triangle when $a = 0.14679$, $b = 0.10433$, $c = 0.04796$.

Ans. $\frac{1}{2} \alpha = 73^\circ 20'.4$; $\frac{1}{2} \beta = 11^\circ 29'.4$; $\frac{1}{2} \gamma = 5^\circ 10'.2$.

100. Case IV. Given Two Sides and the Included Angle (b, c, α). First Method. — *The sum of any two sides of a triangle is to their difference as the tangent of half the sum of the opposite angles is to the tangent of half their difference.* For we have

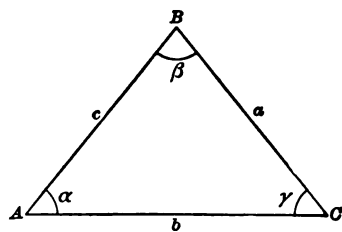


FIG. 61.

$$\frac{b}{c} = \frac{\sin \beta}{\sin \gamma}$$

By composition and division,

$$\frac{b+c}{b-c} = \frac{\sin \beta + \sin \gamma}{\sin \beta - \sin \gamma} = \frac{2 \sin \frac{1}{2}(\beta + \gamma) \cos \frac{1}{2}(\beta - \gamma)}{2 \cos \frac{1}{2}(\beta + \gamma) \sin \frac{1}{2}(\beta - \gamma)} \quad (\text{Art. 73.})$$

$$\therefore \frac{b+c}{b-c} = \frac{\tan \frac{1}{2}(\beta + \gamma)}{\tan \frac{1}{2}(\beta - \gamma)}. \quad (1)$$

But

$$\beta + \gamma = 180^\circ - \alpha; \quad \frac{1}{2}(\beta + \gamma) = 90^\circ - \frac{1}{2}\alpha. \quad \therefore \tan \frac{1}{2}(\beta + \gamma) = \cot \frac{1}{2}\alpha.$$

$$\therefore \tan \frac{1}{2}(\beta - \gamma) = \frac{b-c}{b+c} \cot \frac{1}{2}\alpha. \quad (2)$$

From (2) we find $\frac{1}{2}(\beta - \gamma)$; adding $\frac{1}{2}(\beta - \gamma)$ to $\frac{1}{2}(\beta + \gamma)$, we have β , and subtracting $\frac{1}{2}(\beta - \gamma)$ from $\frac{1}{2}(\beta + \gamma)$, we have γ . Then the third side is found from the sine proportion.

$$\left. \begin{aligned} \text{Formulas: } \tan \frac{1}{2}(\beta - \gamma) &= \frac{b-c}{b+c} \cot \frac{1}{2}\alpha, \\ \frac{1}{2}(\beta + \gamma) &= 90^\circ - \frac{1}{2}\alpha, \\ \beta &= \frac{1}{2}(\beta + \gamma) + \frac{1}{2}(\beta - \gamma), \\ \gamma &= \frac{1}{2}(\beta + \gamma) - \frac{1}{2}(\beta - \gamma), \\ a &= \frac{b \sin \alpha}{\sin \beta} = \frac{c \sin \alpha}{\sin \gamma}. \end{aligned} \right\}$$

In using (1) or (2) the greater side and the greater angle should be written first; thus, if c were greater than b , we should use $c - b$ and $\gamma - \beta$ instead of $b - c$ and $\beta - \gamma$. If the smaller side is written first, the tangent of half the difference of the two angles will be negative, giving the half-difference as an angle between 0° and -90° .

1. Solve the triangle when $b = 0.14367$, $c = 0.11412$, $\alpha = 42^\circ 14'.6$.

$$\therefore \frac{1}{2}\alpha = 21^\circ 7'.3; \quad \frac{1}{2}(\beta + \gamma) = 90^\circ - \frac{1}{2}\alpha = 68^\circ 52'.7.$$

$$\log \tan \frac{1}{2}(\beta - \gamma) = \log(b - c) + \text{col}(b + c) + \log \cot \frac{1}{2}\alpha.$$

$$\log a = \log b + \log \sin \alpha + \text{col} \sin \beta = \log c + \log \sin \alpha + \text{col} \sin \gamma.$$

$$b - c = 0.02955$$

$$\log b = 9.15737$$

$$b + c = 0.25779$$

$$\log \sin \alpha = 9.82755$$

$$\text{col} \sin \beta = 0.00140$$

$$\log(b - c) = 8.47056$$

$$\log a = 8.98632$$

$$\text{col}(b + c) = 0.58874$$

$$a = 0.096900$$

$$\log \cot \frac{1}{2}\alpha = 0.41308$$

$$\log \tan \frac{1}{2}(\beta - \gamma) = 9.47238$$

$$\log c = 9.05737$$

$$\frac{1}{2}(\beta - \gamma) = 16^\circ 31'.7$$

$$\log \sin \alpha = 9.82755$$

$$\frac{1}{2}(\beta + \gamma) = 68^\circ 52'.7$$

$$\text{col} \sin \gamma = 0.10141$$

$$\beta = 85^\circ 24'.4$$

$$\log a = 8.98633$$

$$\gamma = 52^\circ 21'.0$$

$$a = 0.096902$$

2. Solve the triangle when $a = 101.47$, $c = 99.367$, $\beta = 47^\circ 48'.2$.

$$\text{Ans. } \alpha = 67^\circ 27'.1; \quad \gamma = 64^\circ 44'.7; \quad b = 81.396 \text{ or } 81.394.$$

3. Solve the triangle when $b = 19.937$, $c = 62.475$, $\alpha = 130^\circ 9'.4$.

$$\text{Ans. } \beta = 11^\circ 26'.1; \quad \gamma = 38^\circ 24'.5; \quad a = 76.858 \text{ or } 76.860.$$

101. Case IV. Given b , c , α . Second Method.—To prove the equations

$$a \sin \frac{1}{2}(\beta - \gamma) = (b - c) \cos \frac{1}{2}\alpha, \quad \left. \begin{aligned} & \\ & \end{aligned} \right\} \quad (1)$$

$$a \cos \frac{1}{2}(\beta - \gamma) = (b + c) \sin \frac{1}{2}\alpha. \quad \left. \begin{aligned} & \\ & \end{aligned} \right\} \quad (2)$$

$$\begin{aligned}\frac{b}{c} &= \frac{\sin \beta}{\sin \gamma} \quad \therefore \frac{b+c}{c} = \frac{\sin \beta + \sin \gamma}{\sin \gamma} \\ \therefore \frac{b+c}{\sin \beta + \sin \gamma} &= \frac{c}{\sin \gamma} = \frac{a}{\sin \alpha} = \frac{a}{\sin (\beta + \gamma)} \\ \therefore \frac{b+c}{2 \sin \frac{1}{2}(\beta + \gamma) \cos \frac{1}{2}(\beta - \gamma)} &= \frac{a}{2 \sin \frac{1}{2}(\beta + \gamma) \cos \frac{1}{2}(\beta + \gamma)} \\ \therefore a \cos \frac{1}{2}(\beta - \gamma) &= (b+c) \cos \frac{1}{2}(\beta + \gamma) \\ &= (b+c) \sin \frac{1}{2} \alpha. \quad \text{Q.E.D.}\end{aligned}$$

Similarly, $\frac{b-c}{c} = \frac{\sin \beta - \sin \gamma}{\sin \gamma}$ reduces to

$$a \sin \frac{1}{2}(\beta - \gamma) = (b-c) \sin \frac{1}{2}(\beta + \gamma) = (b-c) \cos \frac{1}{2} \alpha.$$

1. Solve the triangle when $b = 0.14367$, $c = 0.11412$, $\alpha = 42^\circ 14'.6$.

$b - c = 0.02955$ (1)	$(4) + (6) = \log [a \sin \frac{1}{2}(\beta - \gamma)] = 8.44036$ (8)
$b + c = 0.25779$ (2)	$\log \sin \frac{1}{2}(\beta - \gamma) = 9.45404$ (12)
$\frac{1}{2} \alpha = 21^\circ 7'.3$ (3)	$\log \cos \frac{1}{2}(\beta - \gamma) = 9.98167$ (12)
	$(5) + (7) = \log [a \cos \frac{1}{2}(\beta - \gamma)] = 8.96799$ (9)
$\log (b - c) = 8.47056$ (4)	$\log \tan \frac{1}{2}(\beta - \gamma) = 9.47237$ (10)
$\log \cos \frac{1}{2} \alpha = 9.96980$ (6)	$\frac{1}{2}(\beta - \gamma) = 16^\circ 31'.7$ (11)
	$\frac{1}{2}(\beta + \gamma) = 68^\circ 52'.7$ (14)
$\log (b + c) = 9.41126$ (5)	$\beta = 85^\circ 24'.4$ (15)
$\log \sin \frac{1}{2} \alpha = 9.55673$ (7)	$\gamma = 52^\circ 21'.0$ (16)
	$(8 - 12) = (9) - (12) = \log a = 8.98632$ (13)
	$a = 0.096900$ (17)

2. Solve the triangle when $b = 2.3671$, $c = 1.4345$, $\alpha = 112^\circ 43'.4$.

$$\text{Ans. } \beta = 42^\circ 54'.5; \gamma = 24^\circ 22'.1; a = 3.2069.$$

3. Solve the triangle when $a = 101.47$, $c = 99.367$, $\beta = 47^\circ 48'.2$.

$$\text{Ans. } \alpha = 67^\circ 27'.1; \gamma = 64^\circ 44'.7; b = 81.396.$$

102. Case IV. Given b, c, α . Third Method.—To find the third side only.

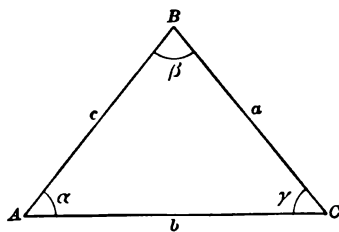


FIG. 62.

$$a^2 = b^2 + c^2 - 2bc \cos \alpha.$$

But

$$\cos \alpha = 1 - 2 \sin^2 \frac{1}{2} \alpha.$$

$$\begin{aligned}\therefore a^2 &= b^2 + c^2 - 2bc + 4bc \sin^2 \frac{1}{2} \alpha \\ &= (b-c)^2 + 4bc \sin^2 \frac{1}{2} \alpha \\ &= (b-c)^2 \left[1 + \frac{4bc \sin^2 \frac{1}{2} \alpha}{(b-c)^2} \right] \\ \therefore a &= (b-c) \sqrt{1 + \frac{4bc \sin^2 \frac{1}{2} \alpha}{(b-c)^2}}\end{aligned}$$

Let x be an angle such that

$$\tan^2 x = \frac{4bc \sin^2 \frac{1}{2} \alpha}{(b-c)^2};$$

$$\text{or} \quad \tan x = \frac{2 \sin \frac{1}{2} \alpha}{b-c} \sqrt{bc}. \quad (1)$$

This assumption is possible, since the value of the second member of (1) must lie between $+\infty$ and $-\infty$, so that there will always be some angle whose tangent is equal to this quantity.

$$\therefore a = (b-c) \sqrt{1 + \tan^2 x} = (b-c) \sec x;$$

$$\text{or} \quad a = \frac{b-c}{\cos x}. \quad (2)$$

First find x from (1), and then a from (2). In these equations $b-c$ is replaced by $c-b$ when $c > b$.

1. Find a when $c = 1.4345$, $b = 2.3671$, and $\alpha = 112^\circ 43'.4$.

$$\log \tan x = \frac{1}{2} (\log b + \log c) + \log 2 + \log \sin \frac{1}{2} \alpha + \text{col } (b-c).$$

$$\log a = \log (b-c) - \log \cos x.$$

$$\log b = 0.37422$$

$$\log c = 0.15670$$

$$\log bc = 0.53092$$

$$\log \sqrt{bc} = 0.26546$$

$$\log 2 = 0.30103$$

$$\log \sin \frac{1}{2} \alpha = 9.92041$$

$$\text{col } (b-c) = 0.03030$$

$$\log \tan x = 0.51720$$

$$x = 73^\circ 5'.6$$

$$\log (b-c) = 9.96970$$

$$-\log \cos x = 9.46361$$

$$\log a = 0.50609$$

$$a = 3.2069$$

2. Find b when $a = 101.47$, $c = 99.367$, $\beta = 47^\circ 48'.2$.

$$\therefore x = 88^\circ 31'.17; b = 81.396.$$

3. Find a when $b = 19.937$, $c = 62.475$, $\alpha = 130^\circ 9'.4$.

$$\therefore x = 56^\circ 23'.7; a = 76.858.$$

OBLIQUE TRIANGLES SOLVED BY RIGHT TRIANGLES.

103. Case I. Given α, a, γ . — In Figs. 63 and 64, on the next page, draw DB perpendicular to AC . Considering the first figure, in the triangle BDC we know a and γ , and we compute DB and DC ; then in the triangle BDA we know DB and α , and we compute AD and c ; then $b = AD + DC$,

completing the solution. In the second figure, where γ is obtuse, we know, in the triangle BDC , a and $DCB = 180^\circ - \gamma$, and we compute DB and CD ; then in the triangle BDA we know DB and α , and we compute c and AD ; then $b = AD - CD$, completing the solution.

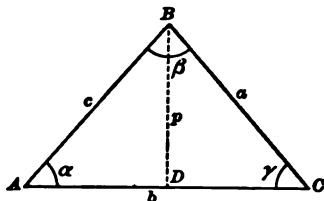


FIG. 63.

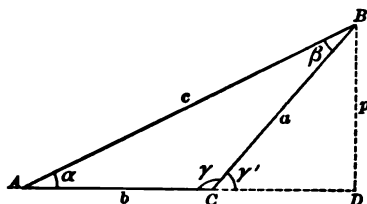


FIG. 64.

1. Solve the triangle when $a = 3.4356$, $\alpha = 17^\circ 43'.4$, $\gamma = 60^\circ 35'.7$.
 $\therefore \beta = 101^\circ 40'.9$; $DC = 1.6868$; $DB = 2.9929$; $AD = 9.3650$;
 $c = 9.8315$; $b = 11.0518$.
2. Solve the triangle when $a = 54.376$, $\gamma = 103^\circ 3'.2$, $\beta = 40^\circ 10'.3$.
Ans. $\alpha = 36^\circ 46'.5$; $c = 88.478$; $b = 58.592$.
3. Solve the triangle when $c = 230.47$, $\alpha = 21^\circ 32'.2$, $\beta = 36^\circ 24'.4$.
Ans. $\gamma = 122^\circ 3'.4$; $a = 99.825$; $b = 161.3975$.

104. Case II. Given a, c, α .—In the right triangle ADB we know c and α , and we compute AD and DB ; then in the triangle BCD we know DB and a , and we find DC and γ ; then

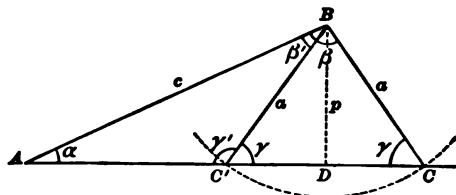


FIG. 65.

$$b = AD + DC; \beta = 180^\circ - (\alpha + \gamma);$$

$$b' = AD - DC; \gamma' = 180^\circ - \gamma; \beta' = 180^\circ - (\alpha + \gamma').$$

Two solutions are possible only when α is acute and a is less than c and greater than DB .

If α is obtuse, as in Fig. 66, we solve first the triangle BAD , then the triangle BCD , and find $b = DC - DA$.

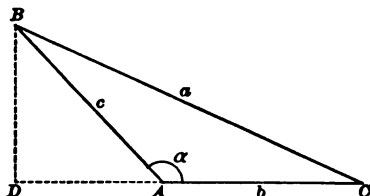


FIG. 66.

1. Solve the triangle when $c = 23.647$, $\alpha = 14.135$, $\alpha = 33^\circ 17'.3$.

$$\therefore AD = 19.767; DB = 12.979; \gamma = 66^\circ 40'.0; DC = 5.5986;$$

$$\text{Ans. } \begin{cases} \gamma = 66^\circ 40'.0; \beta = 80^\circ 2'.7; b = 25.3656; \\ \gamma' = 113^\circ 20'.0; \beta' = 33^\circ 22'.7; b' = 14.1684. \end{cases}$$

2. Solve the triangle when $a = 2.4741$, $c = 1.0003$, $\alpha = 69^\circ 14'.8$.

$$\therefore \gamma = 22^\circ 12'.8; \beta = 88^\circ 32'.4; AD = 0.35445; DC = 2.2905; b = 2.64495.$$

3. Solve the triangle when $a = 10.473$, $b = 12.987$, $\alpha = 44^\circ 11'.3$.

$$\text{Ans. } \begin{cases} \beta = 59^\circ 48'.5; \gamma = 76^\circ 0'.2; c = 14.5793; \\ \beta' = 120^\circ 11'.5; \gamma' = 15^\circ 37'.2; c' = 4.0455. \end{cases}$$

4. Solve the triangle when $a = 0.43477$, $b = 0.40031$, $\alpha = 94^\circ 17'.6$.

$$\text{Ans. } \beta = 66^\circ 39'.6; \gamma = 19^\circ 2'.8; c = 0.142282.$$

105. Case III. Given a , b , c . — In Fig. 63,

$$p^2 = c^2 - AD^2; p^2 = a^2 - DC^2.$$

$$\therefore c^2 - AD^2 = a^2 - DC^2.$$

$$\therefore AD^2 - DC^2 = c^2 - a^2.$$

$$\therefore AD - DC = \frac{(c + a)(c - a)}{AD + DC} = \frac{(c + a)(c - a)}{b},$$

from which $AD - DC$ may be computed. Then

$$AD = \frac{1}{2} [b + (AD - DC)],$$

and

$$DC = \frac{1}{2} [b - (AD - DC)].$$

If either AD or DC is negative, it is exterior to the triangle; that is, the point D is on the line AC produced.

Having found AD and DC , the angles are found from the right triangles DBA and DBC .

1. Solve the triangle when $a = 27.103$, $b = 16.432$, $c = 12.511$.

$$\therefore c - a = -14.592; AD - DC = -35.178; AD = -9.373; DC = 25.805.$$

$$\text{Ans. } \alpha = 138^\circ 31'.2; \gamma = 17^\circ 48'.5; \beta = 23^\circ 40'.3.$$

In this example D lies to the left of A .

2. Solve the triangle when $a = 32.456$, $b = 41.724$, $c = 53.987$.

$$\therefore AD - DC = 44.607; AD = 43.1655; DC = -1.4415.$$

$$\text{Ans. } \alpha = 36^\circ 54'.7; \gamma = 92^\circ 32'.7; \beta = 50^\circ 32'.6.$$

3. Solve the triangle when $a = 0.14679$, $b = 0.10433$, $c = 0.04796$.

$$\therefore AD - DC = -0.18448; AD = -0.040075; DC = +0.144405.$$

$$\text{Ans. } \alpha = 146^\circ 40'.75; \gamma = 10^\circ 21'.0; \beta = 22^\circ 58'.25.$$

106. Case IV. Given b, c, α .—In the triangle ADB , knowing c and α , find AD and DB . Then in the triangle DBC we know DB and $DC = b - AD$, so that we can compute a and γ .

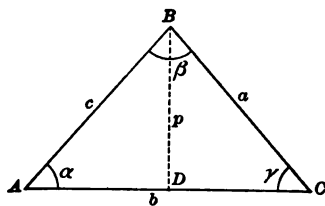


FIG. 67.

1. Solve the triangle when $b = 1143.7$, $c = 1822.4$, $\alpha = 15^\circ 6'.4$.

$$\therefore AD = 1759.5; DB = 474.96; DC = -615.8.$$

$$\text{Ans. } \gamma = 142^\circ 21'.5; a = 777.68; \beta = 22^\circ 32'.1.$$

The negative value of DC shows that D is to the right of C .

2. Solve the triangle when $b = 19.937$, $c = 62.475$, $\alpha = 130^\circ 9'.4$.

$$\therefore AD = -40.288; DC = 60.225.$$

$$\text{Ans. } \gamma = 38^\circ 24'.5; \beta = 11^\circ 26'.1; a = 73.857, \text{ or } 76.858.$$

Note that a is obtuse.

3. Solve the triangle when $a = 101.47$, $c = 99.367$, $\beta = 47^\circ 48'.2$.

$$\text{Ans. } \gamma = 64^\circ 44'.6; \alpha = 67^\circ 27'.2; b = 81.394.$$

AREAS OF TRIANGLES.

107. Given Two Sides and the Included Angle (b, c, α).—Represent the area by A . From geometry, in Fig. 67,

$$A = \frac{1}{2}pb.$$

But

$$p = c \sin \alpha.$$

$$\therefore A = \frac{1}{2}bc \sin \alpha, \quad (1)$$

or, the area of a triangle is equal to half the product of the two sides multiplied by the sine of their included angle.

108. Given One Side and the Three Angles (b, α, β, γ).—Substitute in (1), Art. 107, the value of c found from the sine proportion,

$$c = \frac{b \sin \gamma}{\sin \beta},$$

giving
$$A = \frac{b^2}{2} \cdot \frac{\sin \alpha \sin \gamma}{\sin \beta} \quad (1)$$

109. Given the Three Sides (a, b, c).—We have

$$A = \frac{1}{2} bc \sin \alpha = bc \sin \frac{1}{2} \alpha \cos \frac{1}{2} \alpha.$$

From (2) and (3), Art. 99, we have

$$A = bc \sqrt{\frac{(s-b)(s-c)}{bc}} \sqrt{\frac{s(s-a)}{bc}} = \sqrt{s(s-a)(s-b)(s-c)}. \quad (1)$$

110. Given Two Sides and the Angle Opposite One of them (b, c, β).—First find γ by the formula

$$\sin \gamma = \frac{c}{b} \sin \beta.$$

Then

$$\alpha = 180^\circ - (\beta + \gamma),$$

and

$$A = \frac{1}{2} bc \sin \alpha.$$

EXAMPLES.

1. Find the area when $b = 0.14367$,
 $c = 0.11412$, $\alpha = 42^\circ 14' 6$.

$$\log b = 9.15737$$

$$\log c = 9.05737$$

$$\log \sin \alpha = 9.82755$$

$$\text{col } 2 = 9.69897$$

$$\log A = 7.74126$$

$$A = 0.0055114$$

2. Find the area when $a = 3.4356$,
 $\alpha = 17^\circ 43' 4$, $\gamma = 60^\circ 35' 7$.

$$\therefore \beta = 101^\circ 40' 9.$$

$$\log a^2 = 2 \log a = 1.07200$$

$$\text{col } 2 = 9.69897$$

$$\log \sin \beta = 9.99091$$

$$\log \sin \gamma = 9.94010$$

$$\text{col } \sin \alpha = 0.51652$$

$$\log A = 1.21850$$

$$A = 16.539$$

3. Find the area when $a = 0.0093146$, $b = 0.0176530$, $c = 0.0095768$.

$$2s = 0.0365444$$

$$s = 0.0182722$$

$$s - a = 0.0089576$$

$$s - b = 0.0006192$$

$$s - c = 0.0086954$$

$$\text{sum} = 0.0365444$$

$$a \text{ check.}$$

$$\log s = 8.26179$$

$$\log (s - a) = 7.95219$$

$$\log (s - b) = 6.79183$$

$$\log (s - c) = 7.93929$$

$$2) 10.94510 - 20$$

$$\log A = 5.47265 - 10$$

$$A = 0.000029686$$

4. Find the area when
- $a = 9.4672$
- ,
- $c = 14.433$
- ,
- $\alpha = 11^\circ 14' .3$
- .

$\log c = 1.15936$	$\log a = 0.97622$	$\log a = 0.97622$
$\log \sin \alpha = 9.28979$	$\log c = 1.15936$	$\log c = 1.15936$
$\text{col } \alpha = 9.02378$	$\text{col } 2 = 9.69897$	$\text{col } 2 = 9.69897$
$\log \sin \gamma = 9.47293$	$\log \sin \beta = 9.67899$	$\log \sin \beta' = 9.02259$
$\gamma = 17^\circ 17' .1$	$\log A = 1.51354$	$\log A' = 0.85714$
$\gamma' = 162^\circ 42' .9$	$A = 32.624$	$A' = 7.1968$
$\therefore \beta = 151^\circ 28' .6$		
$\beta' = 6^\circ 2' .8$		

Note that $\log A$ and $\log A'$ can be found by adding $\log \sin \beta$ and $\log \sin \beta'$ respectively to $\log a + \log c + \text{col } 2$, a shorter method than that given in this example.

5. Find the area when
- $a = 0.013456$
- ,
- $b = 0.023678$
- ,
- $\alpha = 40^\circ 31' .4$
- .

Ans. 0.00010351.

6. Find the area when
- $c = 43.145$
- ,
- $\alpha = 40^\circ 40' .3$
- ,
- $\beta = 60^\circ 30' .3$
- . Ans. 538.19.

7. Find the area when
- $a = 1.4142$
- ,
- $b = 1.6735$
- ,
- $c = 2.8533$
- . Ans. 0.83826.

8. Find the area when
- $a = 14.135$
- ,
- $c = 23.647$
- ,
- $\alpha = 33^\circ 17' .3$
- .

Ans. 164.61 or 91.948.

111. Illustrative Examples.—The *bearing* of a line is the angle it makes with the magnetic meridian, shown by the magnetic needle. The letter indicating whether the line is measured north or south of the point of beginning is written, then the number of degrees and minutes in the angle, and then the letter indicating whether the line lies to the east or to the west of the magnetic meridian. Thus, if the bearing of the line AB is S. 60° W., the line is measured from A to the west of south by an angle of 60° .

The distances and the angles given in the examples are horizontal unless otherwise specified.



FIG. 63.

1. From a point on a horizontal plane the angle of elevation to the top of a crag is $40^\circ 28' .6$, and 4163.2 feet farther away in the same vertical plane the angle is $28^\circ 50' .4$. Find the distances from the points to the top of the crag, and its height above the horizontal plane.

$$\therefore BD = 13309 \text{ feet}; AD = 9956.2 \text{ feet}; CD = 6463.0 \text{ feet};$$

$$BC = 11737 \text{ feet}; AC = 7573.2 \text{ feet}.$$

3. A tower 160.43 feet high is situated at the top of a hill (Fig. 69); 600 feet down the hill the angle between the surface of the hill and a line to the top of the tower is $8^{\circ} 40' 4''$. Find the distance to the top of the tower, and the inclination of the ground to a horizontal plane.

$$\therefore \angle ABC = 136^{\circ} 59' 7''; \quad AC = 725.60 \text{ feet}; \quad \angle DAB = 46^{\circ} 59' 7''.$$

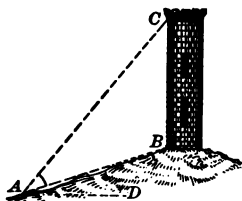


FIG. 69.

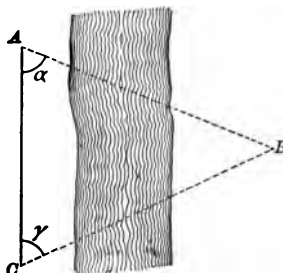


FIG. 70.

3. To find the horizontal distance from a point A to an inaccessible point B (Fig. 70), the horizontal distance AC and the angles α and γ were measured and found to be 1042.3 feet, $72^{\circ} 9' 4''$, and $14^{\circ} 13' 7''$, respectively.

$$\therefore AB = 256.69 \text{ feet}; \quad CB = 994.15 \text{ feet}.$$

4. To find the distance between two points A and B not visible from each other (Fig. 71).—Select a third point C from which A and B are visible, and measure the distances $CA = 444.38$ feet, $CB = 222.76$ feet, and the angle $\angle ACB = 17^{\circ} 17' 6''$.

$$\text{Ans. } AB = 240.97 \text{ feet}.$$

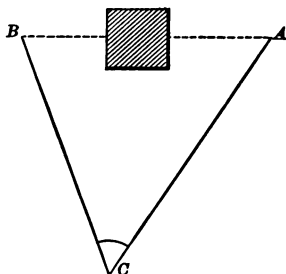


FIG. 71.

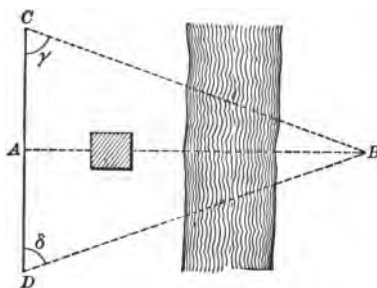


FIG. 72.

5. To find the distance from a point A to another point B , the latter being inaccessible and invisible from A (Fig. 72).—Select two points C and D so that C , A , and D shall be in the same straight line, A and B being visible both from C and from D . From measurement it is found that $CA = 456.72$ feet, $AD = 490.74$ feet, $\gamma = 71^{\circ} 22' 7''$, $\delta = 36^{\circ} 19' 4''$.

$$\therefore CB = 589.10 \text{ feet}; \quad DB = 942.475 \text{ feet}; \quad AB = 619.51, \text{ or } 619.53 \text{ feet}.$$

6. To find the elevation of the top of a church steeple D (Fig. 73) above the horizontal plane ACB , and the distances of the steeple from A and B . — Let the horizontal distance $AB = 435.53$ feet, the horizontal angles $\alpha = 140^\circ 40' 2''$ and $\beta = 10^\circ 7' 6''$, and the vertical angles $\gamma = 32^\circ 45' 6''$ and $\gamma' = 10^\circ 7' 3''$.

$\therefore AC = 156.95$ feet; $BC = 565.74$ feet; $CD = 100.99$, or 101.00 feet.

The agreement of the values of CD is a check upon the observed angles and upon the computations.

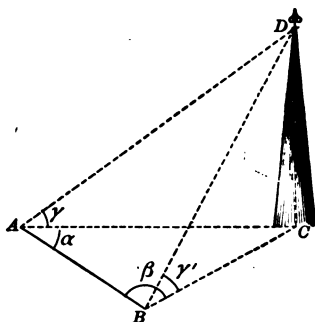


FIG. 73.

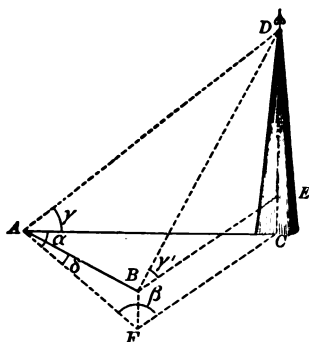


FIG. 74.

7. To find the elevation of the top of a church steeple D (Fig. 74) above the two points A and B , not in the same horizontal plane, the inclined distance from A to B , and its angle of inclination δ to a horizontal plane being measured, as well as the angles α , β , γ , and γ' , shown in the preceding example. — Let $AB = 134.70$ feet, $\delta = 3^\circ 2' 7''$, $\alpha = 43^\circ 14' 8''$, $\beta = 63^\circ 17' 5''$, $\gamma = 56^\circ 36' 6''$, $\gamma' = 62^\circ 17' 3''$.

[First find the horizontal distance AF and the vertical distance FB in the right triangle AFB ; then solve the horizontal triangle AFC ; and then find CD and ED from the right triangles ACD and BED respectively.]

$\therefore AF = 134.51$ feet; $FB = 7.1553$ feet; $FC = BE = 96.135$ feet;

$AC = 125.34$ feet; $CD = 190.17$ feet; $ED = 183.02$ feet.

Check:

$CD = FB + ED$.

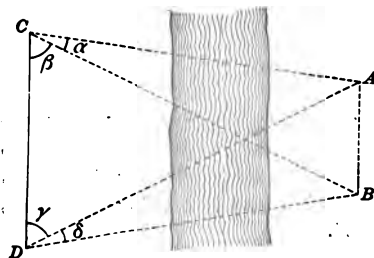


FIG. 75.

8. To find the distance between two inaccessible points A and B . — Select two points C and D from which both A and B can be seen, and measure

$CD = 456.32$ feet, $\alpha = 30^\circ 40' 6''$,

$\beta = 40^\circ 14' 8''$, $\gamma = 35^\circ 16' 4''$,

$\delta = 56^\circ 47' 4''$.

$\therefore AD = 449.09$ feet; $AC = 274.41$ feet;

$BD = 398.66$ feet; $BC = 616.66$ feet;

$AB = 405.57$, or 405.58 feet.

9. To find the distance between two inaccessible points A and B , both being visible from only one accessible point C .—Select a point D from which A and C are visible, and another point E from which B and C are visible. From measurement

$$CD = 943.37 \text{ feet,} \quad CE = 673.33 \text{ feet,}$$

$$\alpha = 72^\circ 9'.3, \quad \beta = 60^\circ 17'.9,$$

$$\gamma = 32^\circ 14'.6, \quad \delta = 67^\circ 33'.9,$$

$$\epsilon = 19^\circ 14'.7.$$

$$\therefore CA = 1217.0 \text{ feet; } CB = 222.28 \text{ feet;}$$

$$AB = 1035.8 \text{ feet.}$$

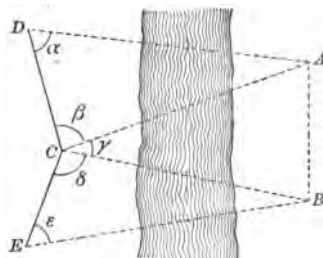


FIG. 76.

10. To find the distance between two inaccessible points A and B , there being no accessible point from which both A and B are visible (Fig. 77).—Select the points C , D , E , and F so that A , C , and E shall be visible from D , and D , F , and B from E . Measure the angles α , β , γ , δ , ϵ , and θ , and the distances CD , DE , and EF . Show how AB may be found from the data thus obtained.

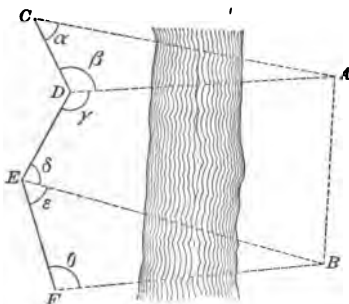


FIG. 77.

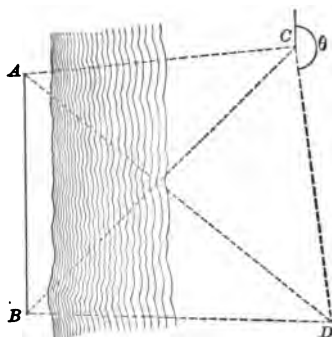


FIG. 78.

11. Two points A and B , 8763.6 feet apart (Fig. 78), are situated at the sea level in the same north and south line; a vessel is seen at C , and an hour later at D . The required quantities are AC , BC , AD , BD , CD , and the angle that CD makes with the north and south line, having measured $BAC = 120^\circ 30'.6$, $BAD = 30^\circ 14'.4$, $ABC = 40^\circ 18'.8$, $ABD = 140^\circ 28'.2$.

$$\therefore AC = 17260 \text{ feet; } BC = 22985 \text{ feet; } AD = 34552.5 \text{ feet; } BD = 27340 \text{ feet;}$$

$$ACD = 63^\circ 14'.5; \quad ADC = 26^\circ 29'.3; \quad BCD = 44^\circ 3'.8; \quad BDC = 35^\circ 46'.8;$$

$$CD = 38696, 38697, \text{ or } 38699 \text{ feet;}$$

$$\theta = 360^\circ - BAC - ACB - BCD = 176^\circ 15'.0,$$

or

$$= ABD + BDA + ADC = 176^\circ 14'.9.$$

12. In measuring the line from A to B , whose direction was known, it was necessary to pass an obstacle at F (Fig. 79). A distance $CD = 144.31$ feet was measured, making an angle $\gamma = 19^\circ 53'.4$ with AB , and the angle $\delta = 140^\circ 10'.3$

was laid off with the transit. It is required to find the distance DE to the line, the distance CE , and the angle ϵ , in order that the line AC may be prolonged.

Ans. $CE = 271.06$ feet; $DE = 143.98$ feet; $\epsilon = 160^\circ 3' 7$.

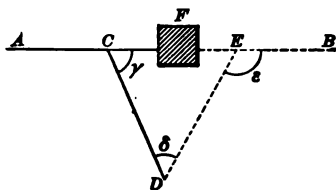


FIG. 79.

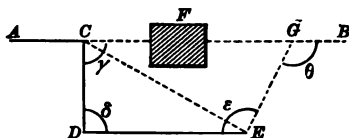


FIG. 80.

13. In passing an obstacle at F it was necessary to use the broken line $CDEG$ (Fig. 80). The distances CD and DE and the angles γ , δ , and ϵ were measured. It is required to find the distance EG to the line AB , the distance CG , and the angle θ , when $CD = 100.37$ feet, $DE = 94.367$ feet, $\gamma = 80^\circ$, $\delta = 101^\circ 19' 8$, and $\epsilon = 110^\circ$.

$\therefore DCE = 37^\circ 53' 3$; $DEC = 40^\circ 46' 9$;
 $CE = 150.67$ feet; $EG = 108.46$ feet;
 $CG = 151.22$ feet; $\theta = 111^\circ 19' 8$.

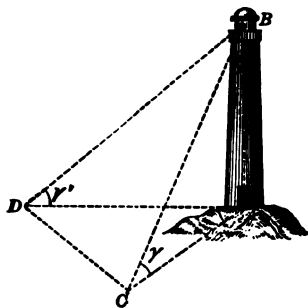


FIG. 81.

14. From the top of a lighthouse AB , 200 feet above the sea level, the angle of depression to a ship was $\gamma = 10^\circ 14' 3$; an hour later it was $\gamma' = 11^\circ 10' 6$; the horizontal angle between the directions of the ship at the two instants was $\alpha = 127^\circ 14' 4$. Find the distance sailed by the ship.

$\therefore AC = 1107.3$ feet; $AD = 1012.2$ feet;
 $CD = 1899.3$ feet.

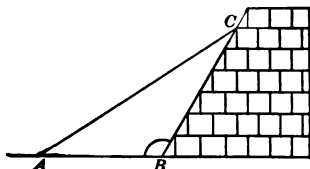


FIG. 82.

15. A ladder 52 feet long is set 20 feet in front of an inclined buttress, and reaches 46 feet up its face. Find the inclination of the face of the buttress.

Ans. $ABC = 95^\circ 51' 8$, or $95^\circ 51' 9$.

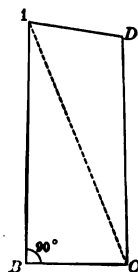


FIG. 83.

16. The sides of a city block measured $AB = 423.24$, $BC = 162.36$, $CD = 420.81$, and $DA = 160.62$ feet, the first two sides being perpendicular to each other. Find the angles between the other sides.

$\therefore AC = 453.31$ feet; $BCA = 69^\circ 0' 8$;
 $BAC = 20^\circ 59' 2$; $ACD = 20^\circ 45' 0$;
 $CAD = 68^\circ 8' 8$; $CDA = 91^\circ 6' 4$;
 $BCD = 89^\circ 45' 8$; $BAD = 89^\circ 8' 0$.

17. A ship B is 12 miles S. 45° W. of a lighthouse A , and sails S. 50° E. to C , a distance of 15 miles. Find its distance from the lighthouse.

Ans. $AC = 18.374$, or 18.375 miles.

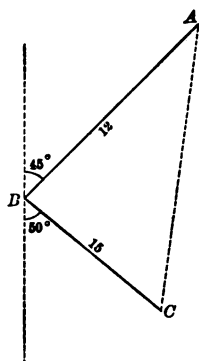


FIG. 84.

18. In surveying a field a thick wood prevents the measurement of the angle ABD and of the distance BD . The angle $ABC = 70^\circ 14'.6$ is measured, a line BC is run 743.86 feet, the angle BCD is found to be $62^\circ 14'.4$, and the distance CD to be 912.82 feet.

$\therefore CBD = 68^\circ 28'.1$; $CDB = 49^\circ 17'.5$;

$BD = 868.34, 868.36$, or 868.38 feet;

$ABD = 138^\circ 42'.7$.

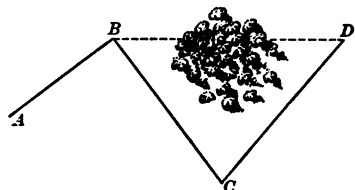


FIG. 85.

19. The distance OE and its bearing $E'OE$ are required, the engineer having measured the distances a, b, c, d , and e and their respective bearings, N. 30° W., S. 60° E., N. 20° E., N. 40° W., and N. 50° E.

$$\begin{aligned} OE' &= OA' - B'A' + B'C' + C'D' + D'E' \\ &= a \cos 30^\circ - b \cos 60^\circ + c \cos 20^\circ \\ &\quad + d \cos 40^\circ + e \cos 50^\circ. \end{aligned}$$

$$\begin{aligned} E'E &= -AA' + B''B + C''C - DD'' + E''E \\ &= -a \sin 30^\circ + b \sin 60^\circ + c \sin 20^\circ \\ &\quad - d \sin 40^\circ + e \sin 50^\circ. \end{aligned}$$

$$\begin{cases} \text{Then } OE \cos E'OE = OE', \\ OE \sin E'OE = E'E; \end{cases}$$

whence OE and $E'OE$ can be found. Then the quadrant of $E'OE$ fixes the direction of the line OE ; thus, if $E'OE = 40^\circ$, the bearing is N. 40° E.; if $E'OE = 110^\circ$, the bearing is S. 70° E.; if $E'OE = 230^\circ$, the bearing is S. 50° W.; if $E'OE = 310^\circ$, the bearing is N. 50° W.

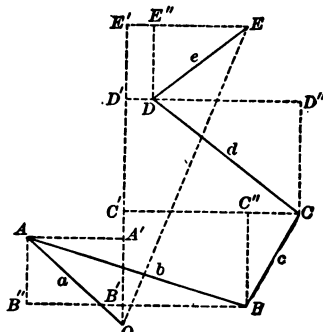


FIG. 86.

20. At a certain point the angles of elevation of the base of a vertical tower and of its top are α and β respectively, the height of the tower being h feet. Prove that the horizontal distance from the point to the tower is

$h \cos \alpha \cos \beta \operatorname{cosec}(\beta - \alpha)$, and that the elevation of its top above the point is $h \cos \alpha \sin \beta \operatorname{cosec}(\beta - \alpha)$.

21. At the top of a vertical tower whose height is h , the angles of depression to two points M and N in the same vertical plane with the tower were α and β respectively ($\beta > \alpha$), the points being in the same horizontal plane with the base of the tower. Prove that the distance MN is $h \sin(\beta - \alpha) \operatorname{cosec} \alpha \operatorname{cosec} \beta$.

22. Two points M and N in a horizontal plane are in the same vertical plane with a tower. The angle of elevation of the top of the tower from M is α , and from N it is β , β being greater than α . Prove that the horizontal distance of the tower from N is $MN \sin \alpha \cos \beta \operatorname{cosec}(\beta - \alpha)$.

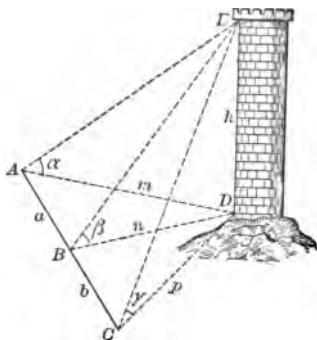


FIG. 87.

23. Three points, A , B , and C , are in the same horizontal line, the distances AB and BC being a and b feet respectively (Fig. 87). The angles of elevation of the top of a tower measured at A , B , and C were α , β , and γ respectively. Find the elevation of the top of the tower above the horizontal plane through the points, and the horizontal distances of the tower from the three points.

$$m = h \cot \alpha; \quad n = h \cot \beta; \quad p = h \cot \gamma;$$

$$m^2 = a^2 + n^2 - 2an \cos \angle ABD;$$

$$p^2 = b^2 + n^2 + 2bn \cos \angle ABD;$$

$$\therefore \frac{a^2 + n^2 - m^2}{2an} = \frac{p^2 - b^2 - n^2}{2bn};$$

$$\therefore h^2 = \frac{ab(a+b)}{a(\cot^2 \gamma - \cot^2 \beta) + b(\cot^2 \alpha - \cot^2 \beta)}$$

24. In Fig. 88 the distances a and b and the angles α , β , and γ are known, and the distance $BC = x$ is required, $ABCD$ being an inaccessible straight line.

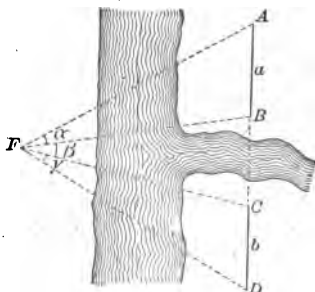


FIG. 88.

$$\frac{a}{\sin \alpha} = \frac{FB}{\sin A}; \quad \frac{a+x}{\sin(\alpha+\beta)} = \frac{FC}{\sin A};$$

$$\therefore \frac{FB}{FC} = \frac{a}{a+x} \cdot \frac{\sin(\alpha+\beta)}{\sin \alpha} \quad (1)$$

$$\frac{b}{\sin \gamma} = \frac{FC}{\sin D}; \quad \frac{b+x}{\sin(\beta+\gamma)} = \frac{FB}{\sin D};$$

$$\therefore \frac{FC}{FB} = \frac{b}{b+x} \cdot \frac{\sin(\beta+\gamma)}{\sin \gamma} \quad (2)$$

Multiplying (1) and (2), we have

$$(a+x)(b+x) \sin \alpha \sin \gamma = ab \sin(\alpha+\beta) \sin(\beta+\gamma),$$

from which x may be found, since the equation is a quadratic in x .

25. Two points A and B in the same vertical plane with the top of a tower are on a sidehill whose angle of inclination to a horizontal plane is δ , the inclined distance AB being a feet. The angles of elevation of the top of the tower were measured at A and B , and found to be α and β . Prove that the horizontal distance of the top of the tower from B is

$$a (\cos \delta \tan \alpha - \sin \delta) \cos \alpha \cos \beta \operatorname{cosec} (\beta - \alpha),$$

and that the elevation of the top above B is

$$a (\cos \delta \tan \alpha - \sin \delta) \cos \alpha \sin \beta \operatorname{cosec} (\beta - \alpha).$$

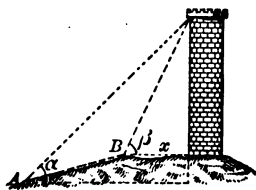


FIG. 89.

26. In a hydrographical survey, the distances between three points, A , B , and C , on the shore having been determined, the observer in the boat P measures the angles δ and ϵ subtended by AB and BC . It is required to find the distances of the boat from the three points.

(1) GRAPHICAL SOLUTION. — Construct on AB the segment of a circle APB that shall contain the measured angle δ , and on BC the segment of a circle BPC that shall contain the angle ϵ . Their point of intersection P will be the position of the boat. There are four possible solutions, only one being shown in the figure.

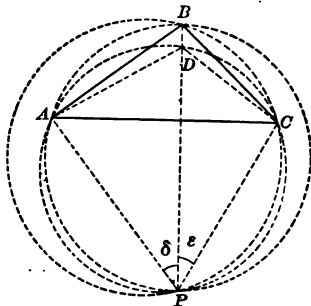


FIG. 90.

(2) ANALYTICAL SOLUTION. — Let $ADCP$ be the circle through A , C , and P . Then $DAC = \epsilon$, and $DCA = \delta$. Hence in the triangle ADC we know one side AC and the three angles; find AD and CD . In the triangle ABC we know the three sides; find the three angles. In the triangle DAB we know two sides and the angle $DAB = CAB - CAD$; find ABD . Then in the triangle ABP we know one side and the three angles; find AP and BP . Also, compute DBC from the triangle DCB , and then BP and CP from the triangle BPC . The values of BP should agree.

In the following examples find the last three elements, the first three being given:

27. $a = 1.0431$, $\beta = 4^\circ 4'.4$, $\gamma = 22^\circ 3'.6$.
 $\therefore a = 153^\circ 52'.0$; $b = 0.16822$; $c = 0.88942$.

28. $a = 103.37$, $\alpha = 10^\circ 11'.3$, $\beta = 83^\circ 43'.6$.
 $\therefore \gamma = 86^\circ 5'.1$; $b = 580.80$; $c = 583.02$.

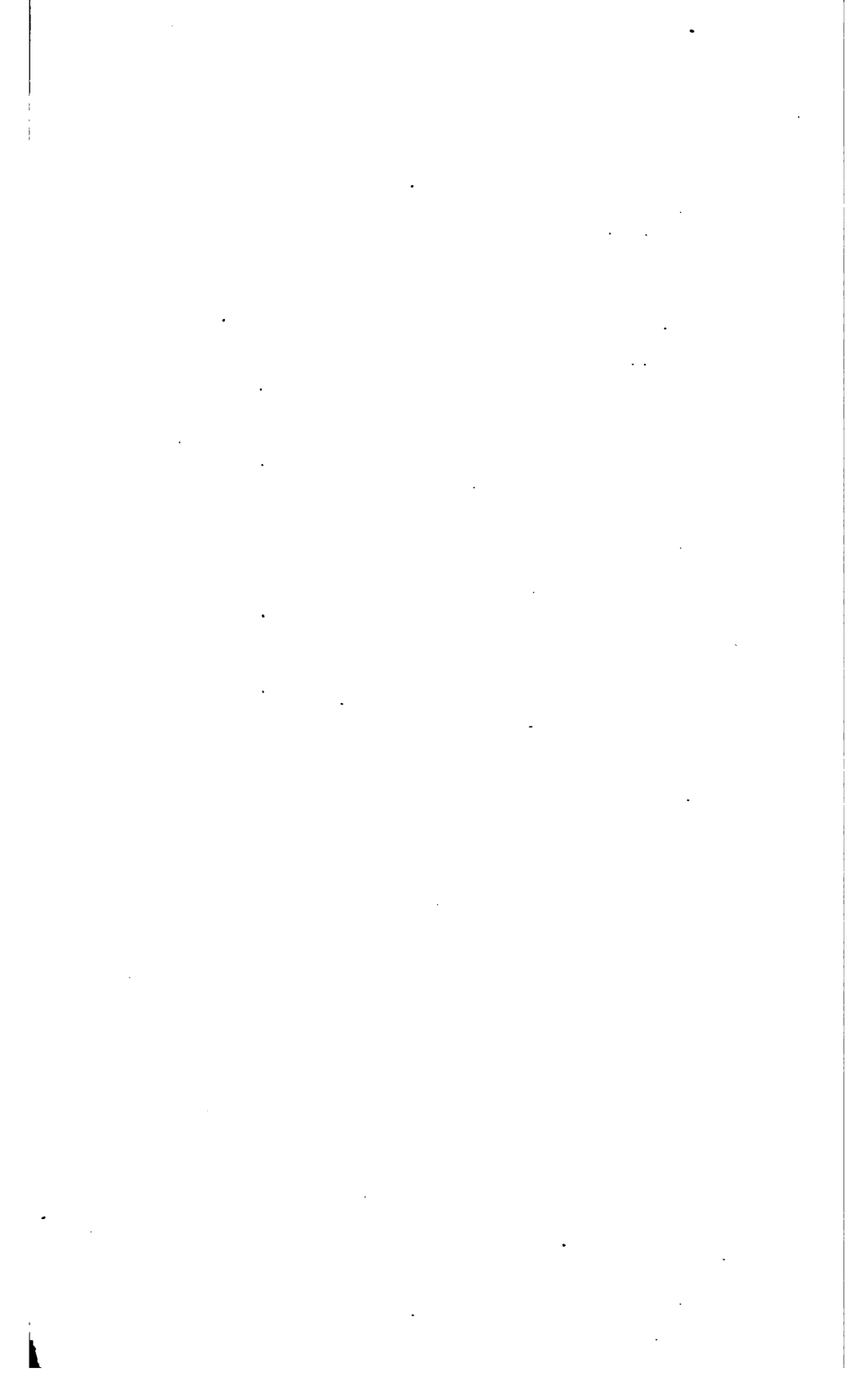
29. $c = 74.344$, $\alpha = 105^\circ 0'.7$, $\beta = 60^\circ 14'.4$.
 $\therefore \gamma = 14^\circ 38'.0$; $a = 283.82$; $b = 255.21$.

30. $c = 0.047365$, $\beta = 40^\circ 7'.7$, $\gamma = 39^\circ 41'.0$.
 $\therefore a = 100^\circ 10'.4$; $a = 0.072990$; $b = 0.047792$.

31. $c = 4.4479$, $\alpha = 11^\circ 11'.3$, $\gamma = 57^\circ 37'.4$.
 $\therefore \beta = 111^\circ 11'.3$; $a = 1.0219$; $b = 4.9106$.

32. $b = 143.97$, $\beta = 30^\circ 36'.8$, $\gamma = 107^\circ 15'.5$.
 $\therefore \alpha = 42^\circ 7'.7$; $a = 189.64$; $c = 269.98$.
33. $b = 10.467$, $c = 1.4321$, $\beta = 114^\circ 10'.3$.
 $\therefore \gamma = 7^\circ 10'.2$; $\alpha = 58^\circ 39'.5$; $a = 9.79875$.
34. $a = 0.67375$, $b = 0.43213$, $\alpha = 147^\circ 11'.3$.
 $\therefore \beta = 20^\circ 20'.2$; $\gamma = 12^\circ 28'.5$; $c = 0.26858$.
35. $a = 1.4742$, $c = 0.97674$, $\alpha = 25^\circ 19'.9$.
 $\therefore \gamma = 16^\circ 28'.1$; $\beta = 138^\circ 12'.0$; $b = 2.2966$.
36. $a = 943.42$, $b = 647.15$, $\alpha = 104^\circ 6'.9$.
 $\therefore \beta = 41^\circ 42'.0$; $\gamma = 34^\circ 11'.1$; $c = 546.59$.
37. $a = 0.10321$, $c = 0.047323$, $\alpha = 45^\circ 9'.7$.
 $\therefore \gamma = 18^\circ 58'.4$; $\beta = 115^\circ 51'.9$; $b = 0.13097$.
38. $a = 4.4321$, $c = 5.4763$, $\gamma = 100^\circ 11'.9$.
 $\therefore \alpha = 52^\circ 48'.1$; $\beta = 27^\circ 0'.0$; $b = 2.5261$.
39. $c = 23.111$, $b = 19.476$, $\gamma = 47^\circ 16'.7$.
 $\therefore \beta = 38^\circ 15'.0$; $\alpha = 94^\circ 28'.3$; $a = 31.363$.
40. $a = 0.11111$, $c = 0.12767$, $\alpha = 23^\circ 15'.6$.
 $\therefore \gamma = 26^\circ 59'.1$; $\beta = 129^\circ 45'.3$; $b = 0.21630$;
 $\gamma' = 153^\circ 0'.9$; $\beta' = 3^\circ 43'.5$; $b' = 0.018279$.
41. $b = 1.4326$, $c = 1.3671$, $\gamma = 44^\circ 17'.3$.
 $\therefore \beta = 47^\circ 1'.9$; $\alpha = 88^\circ 40'.8$; $a = 1.9574$;
 $\beta' = 132^\circ 58'.1$; $\alpha' = 2^\circ 44'.6$; $a' = 0.093706$.
42. $a = 46.703$, $b = 57.147$, $\alpha = 19^\circ 17'.7$.
 $\therefore \beta = 23^\circ 50'.9$; $\gamma = 136^\circ 51'.4$; $c = 96.652$;
 $\beta' = 156^\circ 9'.1$; $\gamma' = 4^\circ 33'.2$; $c' = 11.221$.
43. $a = 9.4327$, $c = 10.4751$, $\alpha = 63^\circ 17'.3$.
 $\therefore \gamma = 82^\circ 45'.0$; $\beta = 33^\circ 57'.7$; $b = 5.8990$;
 $\gamma' = 97^\circ 15'.0$; $\beta' = 19^\circ 27'.7$; $b' = 3.5182$.
44. $a = 0.034337$, $c = 0.062774$, $\alpha = 9^\circ 6'.7$.
 $\therefore \gamma = 16^\circ 49'.7$; $\beta = 154^\circ 3'.6$; $b = 0.094846$;
 $\gamma' = 163^\circ 10'.3$; $\beta' = 7^\circ 43'.0$; $b' = 0.029115$.
45. $a = 0.79797$, $b = 0.46731$, $\beta = 23^\circ 19'.6$.
 $\therefore \alpha = 42^\circ 32'.5$; $\gamma = 114^\circ 7'.9$; $c = 1.07705$;
 $\alpha' = 137^\circ 27'.5$; $\gamma' = 19^\circ 12'.9$; $c' = 0.38841$.
46. $a = 37.456$, $b = 43.987$, $c = 13.498$.
 $\therefore \frac{1}{2}\alpha = 26^\circ 31'.0$; $\frac{1}{2}\beta = 55^\circ 7'.0$; $\frac{1}{2}\gamma = 8^\circ 22'.0$.
47. $a = 2.4568$, $b = 2.4743$, $c = 1.0047$.
 $\therefore \frac{1}{2}\alpha = 38^\circ 38'.0$; $\frac{1}{2}\beta = 39^\circ 36'.7$; $\frac{1}{2}\gamma = 11^\circ 45'.3$.
48. $a = 47.474$, $b = 100.980$, $c = 93.929$.
 $\therefore \frac{1}{2}\alpha = 13^\circ 56'.8$; $\frac{1}{2}\beta = 42^\circ 10'.2$; $\frac{1}{2}\gamma = 33^\circ 53'.0$.
49. $a = 14.567$, $b = 9.4769$, $c = 11.113$.
 $\therefore \frac{1}{2}\alpha = 44^\circ 50'.9$; $\frac{1}{2}\beta = 20^\circ 17'.5$; $\frac{1}{2}\gamma = 24^\circ 51'.5$.

50. $a = 2.1476$, $b = 1.9397$, $c = 3.4345$.
 $\therefore \frac{1}{2}a = 17^\circ 22'.8$; $\frac{1}{2}\beta = 15^\circ 29'.8$; $\frac{1}{2}\gamma = 57^\circ 7'.3$.
51. $a = 115.03$, $b = 129.15$, $c = 112.06$.
 $\therefore \frac{1}{2}a = 28^\circ 12'.9$; $\frac{1}{2}\beta = 34^\circ 39'.2$; $\frac{1}{2}\gamma = 27^\circ 7'.9$.
52. $b = 113.47$, $c = 227.79$, $a = 19^\circ 43'.4$.
 $\therefore \beta = 17^\circ 33'.8$; $\gamma = 142^\circ 42'.8$; $a = 126.90$;
or $\log \tan x = 9.68278$; $a = 126.89$.
53. $a = 99.416$, $c = 90.432$, $\beta = 11^\circ 7'.8$.
 $\therefore a = 110^\circ 20'.4$; $\gamma = 58^\circ 31'.8$; $b = 20.467$;
or $\log \tan x = 0.31110$; $b = 20.467$.
54. $a = 1.4342$, $b = 9.7672$; $\gamma = 109^\circ 19'.6$.
 $\therefore a = 7^\circ 31'.7$; $\beta = 63^\circ 8'.7$; $c = 10.330$, or 10.331 ;
or $\log \tan x = 9.86498$; $c = 10.331$.
55. $a = 1003.7$, $b = 943.67$, $\gamma = 101^\circ 19'.8$.
 $\therefore a = 40^\circ 46'.9$; $\beta = 37^\circ 53'.3$; $c = 1506.7$;
or $\log \tan x = 1.39930$; $c = 1506.7$.
56. $a = 222.76$, $b = 444.38$, $\gamma = 17^\circ 17'.6$.
 $\therefore a = 15^\circ 57'.0$; $\beta = 146^\circ 45'.4$; $c = 240.97$;
or $\log \tan x = 9.63029$; $c = 240.97$.
57. $a = 363.24$, $b = 146.18$, $\gamma = 68^\circ 14'.4$.
 $\therefore a = 88^\circ 2'.6$; $\beta = 23^\circ 43'.0$; $c = 337.55$, or 337.56 ;
or $\log \tan x = 0.07590$; $c = 337.55$.



PART TWO.

SPHERICAL TRIGONOMETRY.



CHAPTER VIII.

DEFINITIONS AND CONSTRUCTIONS.

112. Spherical Trigonometry treats of the relations between the face angles and the edge angles of a trihedral angle.

An *edge angle* is the angle between two of the three planes forming the trihedral angle; it is measured by the angle between the lines cut from the two planes by a plane perpendicular to the edge in which the two planes intersect.

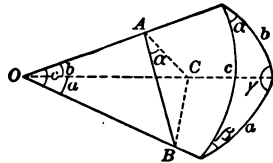


FIG. 91.

A *face angle* is the angle between two of the edges.

113. Representation of Trihedral Angles. — The relations between the elements of a trihedral angle are discussed by means of the spherical triangle formed by the intersections of the faces with a sphere described with any radius about the vertex as a center. The faces will cut arcs of great circles from the surface of the sphere, their angular measures being the same as those of the face angles; and the angles of the spherical triangle will correspond to the edge angles, each being measured by the angle between two lines lying in the planes of the faces and perpendicular to the line of intersection of the faces.

Hence, in the spherical triangle the sides correspond to the face angles, and the angles to the edge angles of the trihedral angle.

The lengths of the sides in linear measure will depend upon the radius of the sphere, and are computed, when the radius is known, by the proportion

$$360^\circ : \alpha = 2\pi r : l, \quad (1)$$

where α is the number of degrees in the arc, and l is its length.

114. Limitation of Values. — We shall consider only those triangles in which each element is less than 180° . In the general spherical triangle the sides and angles may have values greater than 180° , but in such a case it is always possible to substitute for the triangle, in the computations, another in which each element shall be less than 180° .

115. Definitions and Relations. — A *great circle* is cut from the surface of a sphere by a plane passing through its center; its radius is equal to the radius of the sphere.

A *small circle* is cut from the surface by a plane not passing through the center; its radius is always less than the radius of the sphere.

Two planes passing through the center will intersect in a diameter of the sphere, and the two corresponding great circles will intersect at the ends of this diameter. Hence any two great circles will intersect at two points 180° apart.

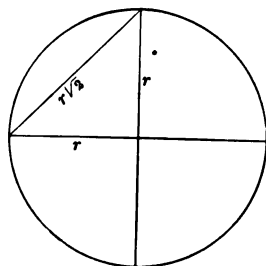


FIG. 92.

To describe a great circle on a sphere, separate the points of a pair of compasses by a distance equal to the chord of 90° , or $r\sqrt{2}$, and describe an arc about any point. If any other distance is used, a small circle will be described. The point used as the

center is called the *pole* of the great circle; its distance from all points on the great circle is evidently 90° .

Any great circle passing through the pole of another great circle will be perpendicular to that great circle

Any two great circles drawn perpendicular to a third great circle will intersect in its pole.

A great circle perpendicular to two great circles will pass through the poles of both, and its plane will be perpendicular to the diameter joining the points of intersection of the two great circles.

The angle between two arcs of great circles is measured by the arc of a great circle described about the vertex as a pole, and limited by the sides, produced if necessary.

The shortest distance between two points on a sphere is the arc of the great circle passing through the points.

116. Constructions. — To find the pole of a given great circle: from any two points on the circle as poles, describe arcs of great circles, and their intersection will be the point required.

To draw a great circle through two points: find the pole as before, and describe the great circle.

To draw a great circle through a given point perpendicular to a given great circle: from the point as a pole describe an arc of a great circle; its point of intersection with the given circle will be the pole of the required circle. Or, find the pole of the given circle, and then draw the great circle through this pole and the given point.

To cut from a great circle an arc n° long: separate the points of the compasses by a distance equal to the chord of n° , or $2r \sin \frac{1}{2}n^\circ$, place the points on the great circle, and the arc intercepted will be the one required.

To construct a great circle passing through a given point and making a given angle with a given great circle: in Fig. 93, let ACB be the given great circle,* P its pole, F the given point, and α the given angle. With P as a pole, draw the small circle $P'P''$ such that the angular

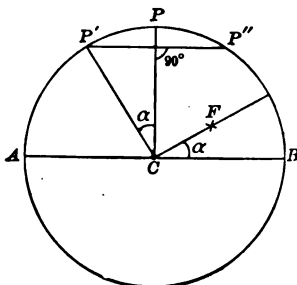


FIG. 93.

* The planes of the great circles ACB and CF , and of the small circle $P'P''$, are perpendicular to the paper.

distance $PP' = \alpha$; then the pole of the required great circle must be on this small circle. With F as a pole, describe an arc of a great circle cutting the small circle $P'P''$ in two points; these points will be the poles of two great circles through F , both of which satisfy the given conditions. Only the great circle CF , whose pole is P' , is shown in the figure.

To construct a great circle making a given angle with a given great circle, the point of intersection being given: from the given point as a pole describe a great circle, lay off on it from the given circle a distance equal in angular measure to the given angle, and pass a great circle through the point thus found and the given point of intersection.

117. Definitions. — A right spherical triangle is one which has one angle equal to 90° ; a birectangular triangle has two angles each equal to 90° ; a trirectangular triangle has three angles each equal to 90° .

A quadrantal triangle has one side equal to a quadrant, or 90° ; a biquadrantal triangle has two sides each equal to a quadrant; a triquadrantal triangle has three sides each equal to a quadrant.

A birectangular triangle is also biquadrantal, and a trirectangular triangle is also triquadrantal; and *vice versa*.

118. The Polar Triangle of any triangle is constructed by describing arcs of great circles about the vertices of the original triangle as poles.

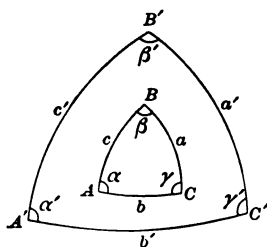


FIG. 94.

Thus, about A , B , and C as poles, describe the arcs $B'C'$, $A'C'$, and $A'B'$, respectively; that triangle is called the polar in which the vertices A and A' , B and B' , C and C' are on the same side of BC , AC , and AB , respectively.

The vertices of the polar triangle will be the poles of the sides of the original triangle, so that either triangle will be the polar of the other.

The sides of a triangle are the supplements of the opposite angles of the polar, and the angles are the supplements of the

opposite sides of the polar; $a' = 180^\circ - a$, $a' = 180^\circ - a$. Thus, if the angles of a triangle be 120° , 80° , and 60° , the opposite sides of the polar will be 60° , 100° , and 120° .

The polar of a quadrantal triangle is a right triangle, the angle in the polar opposite the quadrant being equal to the supplement of 90° ; the polar of a biquadrantal triangle is birectangular; the polar of a triquadrantal triangle is trirectangular; and *vice versa*.

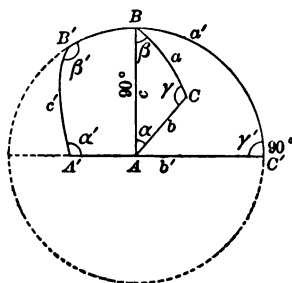


FIG. 95.

The triquadrantal triangle is its own polar, each vertex being the pole of the opposite side.

119. In Any Spherical Triangle :

- (1) Each side must be an arc of a great circle.
- (2) Each side must be less than the sum of the other two.
- (3) The greater side is opposite the greater angle, and conversely. Equal sides are opposite equal angles.
- (4) The sum of the sides must be less than 360° .
- (5) The sum of the angles must be greater than 180° and less than 540° .

120. Construction of Triangles. — (1) Given the three sides, a , b , c . — Draw an arc of a great circle and lay off on it an arc equal to one of the sides, as a . From the extremities of this arc as poles, with radii equal to the chords of b and c respectively, describe arcs of small circles with the compasses, and find their point of intersection. Join this point and the extremities of a by arcs of great circles, and the triangle will be constructed.

(2) Given the three angles, α , β , γ . — Find the sides of the polar triangle, construct it, and then construct the given triangle by using the vertices of the polar as poles.

(3) Given two sides and the included angle, a , b , γ . — Draw an arc of a great circle, and lay off on it an arc equal to one of the sides, as a . Pass an arc of a great circle through one extremity of a , making the angle γ with a , and lay off on it an arc equal to b . Join the extremities of a and b by an arc of a great circle, and the triangle will be constructed.

(4) Given two angles and their included side, α, β, c . — In the polar we know two sides and the included angle, and hence we can construct it by the method just given. Having the polar, we can then construct the required triangle.

Or, draw a great circle and lay off on it an arc equal to c ; at the extremities of this arc, construct arcs of great circles making the angles α and β with c ; their point of intersection will be the third vertex.

(5) Given two sides and the angle opposite one of them, a, b, α . — Draw any great circle ADA' , and through any point

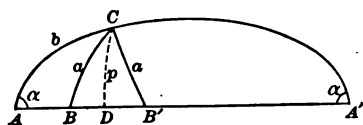


FIG. 96.

on it, as A , draw a great circle making the angle $DAC = \alpha$ with it. On this circle lay off from A the distance $AC = b$. With C as a pole, describe a small circle whose radius is equal to

the chord of a , using the compasses; pass arcs of great circles through C , and the points B and B' where this small circle intersects the first great circle ADA' , and the triangle will be constructed.

There will be, in general, two points of intersection, and there may therefore be two triangles that will satisfy the conditions of the problem. Only those triangles can be taken in which each side is less than 180° , *i.e.* both B and B' must lie on the arc ADA' between A and A' , these points being 180° apart.

If α is acute, as in Fig. 96, a must be greater than p and less than the shorter of the two distances CA and CA' (b and $180^\circ - b$) in order that there may be two solutions.

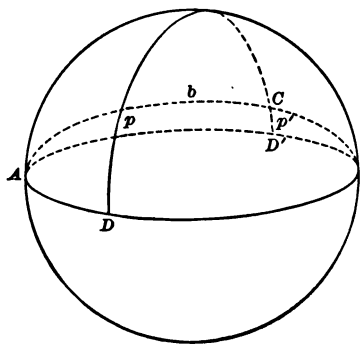


FIG. 97.

If α is obtuse, as in Fig. 97, CD' is the least and CD the greatest distance of C from $ADA'D'$, DCD' being perpendicular to $ADA'D'$. Therefore a must be less than p , in order that the small circle may cut $ADA'D'$; a must also be greater

than the longer of the two distances CA and CA' (b and $180^\circ - b$) in order that the two points of intersection may fall on the arc ADA' .

The conditions, therefore, for two solutions are :

$$\alpha \text{ acute : } a > p, a < b, a < 180^\circ - b.$$

$$\alpha \text{ obtuse : } a < p, a > b, a > 180^\circ - b.$$

Or, a must be intermediate in value between p and both b and $180^\circ - b$.

If a is intermediate in value only between p and either b or $180^\circ - b$, there will be one solution.

If a is not intermediate in value between p and either b or $180^\circ - b$, no solution will be possible, but if $p = a$, there will be one solution — a right triangle.

(6) Given two angles and the side opposite one of them, α, β, a . — In the polar triangle we know two sides and the angle opposite one of them, and we can construct it; having the polar we can construct the required triangle.

As the polar triangle may admit of two solutions, there may be two solutions of the problem.

CHAPTER IX.

GENERAL FORMULAS.

121. The Cosine of Any Side of a Spherical Triangle is equal to the product of the cosines of the other two sides, increased by the product of the sines of these two sides multiplied by the cosine of their included angle. — Let the plane BAC be perpendicular to OA at any point A , and let AC , BC , and BA be its intersections with the faces of the trihedral angle.

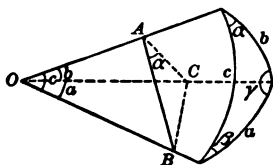


FIG. 98.

Then $BAC = \alpha$, and AB and AC are perpendicular to OA , i.e. OAB and OAC are triangles right-angled at A .

In the triangle BAC we have

$$BC^2 = AB^2 + AC^2 - 2 AB \cdot AC \cos \alpha.$$

In the triangle BOC ,

$$BC^2 = OB^2 + OC^2 - 2 OB \cdot OC \cos a.$$

Equating the values of BC^2 , and transposing,

$$2 OB \cdot OC \cos a = OB^2 - AB^2 + OC^2 - AC^2 + 2 AB \cdot AC \cos \alpha.$$

In the right triangles OAB and OAC ,

$$OB^2 - AB^2 = OA^2, \text{ and } OC^2 - AC^2 = OA^2.$$

$$\therefore 2 OB \cdot OC \cos a = OA^2 + OA^2 + 2 AB \cdot AC \cos \alpha;$$

or
$$OB \cdot OC \cos a = OA^2 + AB \cdot AC \cos \alpha.$$

$$\therefore \cos a = \frac{OA}{OC} \cdot \frac{OA}{OB} + \frac{AC}{OC} \cdot \frac{AB}{OB} \cos \alpha;$$

or
$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

(1)

In this proof b and c are assumed to be less than 90° , while a and α may have any values less than 180° . The formula is true, however, when either b or c , or both b and c , exceed 90° .

If, in the triangle represented by the full lines (Fig. 99), b is greater than 90° , then in the dotted triangle formed by completing the arcs of great circles, the two sides are $180^\circ - b$, and c , both less than 90° , and the other side and its opposite angle are $180^\circ - a$, and $180^\circ - \alpha$. Hence we can apply (1) to the dotted triangle, giving

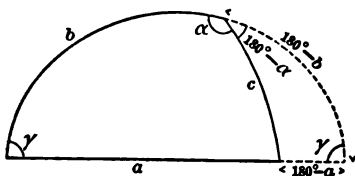


FIG. 99.

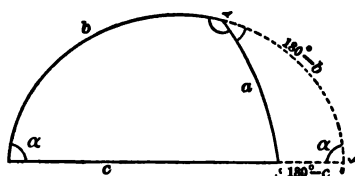


FIG. 100.

$$\cos (180^\circ - a) = \cos (180^\circ - b) \cos c + \sin (180^\circ - b) \sin c \cos (180^\circ - \alpha).$$

$$\therefore -\cos a = -\cos b \cos c - \sin b \sin c \cos \alpha.$$

$$\therefore \cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

Q.E.D.

If both b and c are greater than 90° , as in Fig. 100, then in the dotted triangle the two sides are $180^\circ - b$, and $180^\circ - c$, and the other side and its opposite angle are a and α .

$$\begin{aligned} \therefore \cos a &= \cos (180^\circ - b) \cos (180^\circ - c) + \sin (180^\circ - b) \sin (180^\circ - c) \cos \alpha \\ &= (-\cos b)(-\cos c) + \sin b \sin c \cos \alpha. \end{aligned}$$

$$\therefore \cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

Q.E.D.

Therefore the formula is always true when each of the elements of the triangle is less than 180° .

No assumption, then, has been made concerning any element that is not true for all the others. We may therefore change any angle to another, as α to β , if at the same time we change the sides opposite, as a to b , making also the reverse changes, b to a and β to α , in the formula; for this is equivalent to changing the *names* arbitrarily assigned to the sides and angles. Thus, to permute (1) to find $\cos c$, we change a to c , α to γ , c to a , and γ to α , if they occur in the formula, while b and β will not be affected.

$$\therefore \cos c = \cos b \cos a + \sin b \sin a \cos \gamma.$$

If we assume that our triangle is right-angled, γ being equal to 90° , we can permute between a and b , and α and β , since no assumption is made concerning a and α that is not equally true concerning b and β . But we cannot permute

between α and γ , because γ is assumed to be equal to 90° , while no such assumption is made concerning α .

Permuting (1) in the oblique-angled triangle, we have

$$\left. \begin{aligned} \cos \alpha &= \cos b \cos c + \sin b \sin c \cos \alpha, \\ \cos b &= \cos \alpha \cos c + \sin \alpha \sin c \cos \beta, \\ \cos c &= \cos \alpha \cos b + \sin \alpha \sin b \cos \gamma. \end{aligned} \right\} \quad (2)$$

Eq. (1) is called the fundamental equation of spherical trigonometry, since all the other formulas may be derived from it.

122. The Cosine of Any Angle of a Spherical Triangle is equal to the product of the sines of the other two angles multiplied by the cosine of their included side, diminished by the product of the cosines of the other two angles. — We have

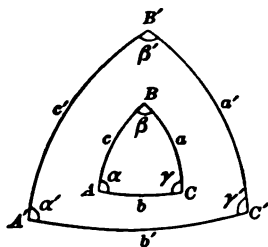


FIG. 101.

$$\cos \alpha = \cos b \cos c + \sin b \sin c \cos \alpha. \quad (1)$$

Since the angles of the polar triangle are the supplements of the sides opposite in the original triangle, and *vice versa*, we have

$$a = 180^\circ - a', \quad b = 180^\circ - \beta', \quad c = 180^\circ - \gamma', \quad \alpha = 180^\circ - \alpha'.$$

Substituting in (1),

$$\begin{aligned} \cos (180^\circ - \alpha') &= \cos (180^\circ - \beta') \cos (180^\circ - \gamma') \\ &\quad + \sin (180^\circ - \beta') \sin (180^\circ - \gamma') \cos (180^\circ - \alpha'), \end{aligned}$$

$$\text{or} \quad -\cos \alpha' = (-\cos \beta')(-\cos \gamma') + \sin \beta' \sin \gamma' (-\cos \alpha').$$

$$\therefore \cos \alpha' = -\cos \beta' \cos \gamma' + \sin \beta' \sin \gamma' \cos \alpha'.$$

This formula expresses a relation between the elements of the polar triangle; but, since the polar may be *any* spherical triangle, it expresses the value of the cosine of an angle of *any* spherical triangle. Dropping the primes and permuting,

$$\left. \begin{aligned} \cos \alpha &= -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos \alpha, \\ \cos \beta &= -\cos \alpha \cos \gamma + \sin \alpha \sin \gamma \cos b, \\ \cos \gamma &= -\cos \alpha \cos b + \sin \alpha \sin b \cos c. \end{aligned} \right\} \quad (2)$$

123. The Sine Proportion. — *The sines of the sides of a spherical triangle are to each other as the sines of the opposite angles.*

*First Proof.** — From (1), Art. 121,

$$\sin b \sin c \cos a = \cos a - \cos b \cos c.$$

$$\therefore \sin^2 b \sin^2 c \cos^2 a = \cos^2 a + \cos^2 b \cos^2 c - 2 \cos a \cos b \cos c.$$

$$\therefore \sin^2 b \sin^2 c (1 - \sin^2 a) = \cos^2 a + \cos^2 b \cos^2 c - 2 \cos a \cos b \cos c.$$

$$\therefore \sin^2 b \sin^2 c \sin^2 a = \sin^2 b \sin^2 c - \cos^2 a - \cos^2 b \cos^2 c + 2 \cos a \cos b \cos c$$

$$= (1 - \cos^2 b)(1 - \cos^2 c) - \cos^2 a - \cos^2 b \cos^2 c + 2 \cos a \cos b \cos c \\ = 1 - \cos^2 b - \cos^2 c - \cos^2 a + 2 \cos a \cos b \cos c.$$

Dividing both sides by $\sin^2 a \sin^2 b \sin^2 c$,

$$\frac{\sin^2 a}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}.$$

Permuting,

$$\frac{\sin^2 b}{\sin^2 b} = \frac{1 - \cos^2 b - \cos^2 a - \cos^2 c + 2 \cos b \cos a \cos c}{\sin^2 b \sin^2 a \sin^2 c},$$

$$\frac{\sin^2 c}{\sin^2 c} = \frac{1 - \cos^2 c - \cos^2 b - \cos^2 a + 2 \cos c \cos b \cos a}{\sin^2 c \sin^2 b \sin^2 a}.$$

The second members of the three equations are identical.

$$\therefore \frac{\sin^2 a}{\sin^2 a} = \frac{\sin^2 b}{\sin^2 b} = \frac{\sin^2 c}{\sin^2 c},$$

or

$$\frac{\sin a}{\sin a} = \frac{\sin b}{\sin b} = \frac{\sin c}{\sin c}. \quad (1)$$

Second Proof. — From any point A on OA pass the planes ADC and ADB perpendicular to OC and OB , respectively, and let AD be their line of intersection. Then AD will be perpendicular to the plane BOC , being the intersection of two planes perpendicular to BOC , and DB and DC will be perpendicular to OB and OC , respectively. The triangles ADC and ADB will be right-angled at D , and

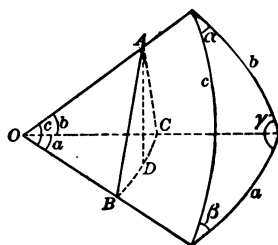


FIG. 102.

* Or find $\cos a$ from (1), Art. 121; then $\sin^2 a = 1 - \cos^2 a$, etc.

the triangles ACO and ABO will be right-angled at C and B .

Also $ABD = \beta$ and $ACD = \gamma$. Then

$$AD = AB \sin ABD = AB \sin \beta,$$

$$\text{and } AD = AC \sin ACD = AC \sin \gamma.$$

$$\therefore AB \sin \beta = AC \sin \gamma$$

$$\text{But } AB = OA \sin c,$$

$$\text{and } AC = OA \sin b.$$

$$\therefore OA \sin c \sin \beta = OA \sin b \sin \gamma.$$

$$\therefore \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c}.$$

Permuting,
$$\frac{\sin \alpha}{\sin a} = \frac{\sin \beta}{\sin b} = \frac{\sin \gamma}{\sin c} \quad (1)$$

124. Additional Formulas. — We have

$$\cos b = \cos a \cos c + \sin a \sin c \cos \beta.$$

$$\begin{aligned} \therefore \sin a \sin c \cos \beta &= \cos b - (\cos b \cos c + \sin b \sin c \cos \alpha) \cos c \\ &= \cos b - \cos b \cos^2 c - \sin b \sin c \cos c \cos \alpha \\ &= \cos b \sin^2 c - \sin b \sin c \cos c \cos \alpha. \end{aligned}$$

$$\therefore \sin a \cos \beta = \cos b \sin c - \sin b \cos c \cos \alpha. \quad (1)$$

Applying (1) to the polar triangle and dropping the primes,

$$\sin \alpha \cos b = \cos \beta \sin \gamma + \sin \beta \cos \gamma \cos \alpha. \quad (2)$$

Dividing (1), member for member, by the equation

$$\sin a \sin \beta = \sin b \sin \alpha,$$

we have
$$\cot \beta = \frac{\cot b \sin c - \cos c \cos \alpha}{\sin \alpha};$$

$$\therefore \sin \alpha \cot \beta = \cot b \sin c - \cos c \cos \alpha. \quad (3)$$

Transposing,

$$\sin c \cot b = \sin \alpha \cot \beta + \cos c \cos \alpha.$$

Permuting,

$$\sin \alpha \cot b = \cot \beta \sin \gamma + \cos \alpha \cos \gamma. \quad (4)$$

Other formulas may be found by permuting (1), (2), and (3). Among these are the following :

from (1), $\sin \alpha \cos \gamma = \cos c \sin b - \sin c \cos b \cos \alpha; \quad (5)$

from (3), $\sin \alpha \cot \gamma = \cot c \sin b - \cos b \cos \alpha, \quad (6)$

and $\sin \gamma \cot \alpha = \cot a \sin b - \cos b \cos \gamma. \quad (7)$

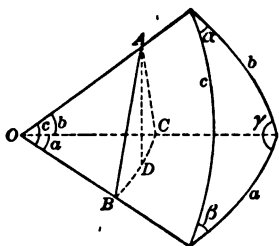


Fig. 108.

CHAPTER X.

RIGHT SPHERICAL TRIANGLES.

125. Formulas for Right Spherical Triangles. — The following equations have been shown in Chap. IX to be true for all spherical triangles :

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma, \quad (a)$$

$$\cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a, \quad (b)$$

$$\cos \gamma = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c, \quad (c)$$

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}, \quad (d)$$

$$\sin a \cos \gamma = \cos c \sin b - \sin c \cos b \cos \alpha, \quad (e)$$

$$\sin \gamma \cot \alpha = \cot a \sin b - \cos b \cos \gamma. \quad (f)$$

By making $\gamma = 90^\circ$ we get seven formulas applicable to right triangles, and by permuting these three others are found.

$$\text{From (a),} \quad \cos c = \cos a \cos b. \quad (1)$$

$$\text{From (c),} \quad \cos c = \cot \alpha \cot \beta. \quad (2)$$

$$\begin{array}{l} \text{From (b),} \\ \text{Permuting,} \end{array} \quad \left. \begin{array}{l} \cos \alpha = \sin \beta \cos a. \\ \cos \beta = \sin \alpha \cos b. \end{array} \right\} \quad (3)$$

$$\begin{array}{l} \text{From (e),} \\ \text{Permuting,} \end{array} \quad \left. \begin{array}{l} \cos \alpha = \tan b \cot c. \\ \cos \beta = \tan a \cot c. \end{array} \right\} \quad (4)$$

$$\begin{array}{l} \text{From (d),} \\ \text{From (d),} \end{array} \quad \left. \begin{array}{l} \sin a = \sin c \sin \alpha. \\ \sin b = \sin c \sin \beta. \end{array} \right\} \quad (5)$$

$$\begin{array}{l} \text{From (f),} \\ \text{Permuting,} \end{array} \quad \left. \begin{array}{l} \sin b = \tan a \cot \alpha. \\ \sin a = \tan b \cot \beta. \end{array} \right\} \quad (6)$$

126. Formulas for Right Spherical Triangles. Geometrical Proof.—Let OB be unity. From B pass the plane BAC perpendicular to OA . Then AB and AC are perpendicular to OA , and CB is perpendicular to OC .

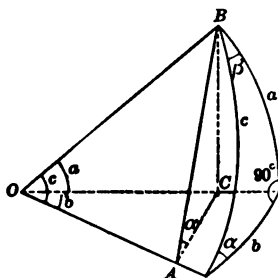


FIG. 104.

$$\therefore CB = \sin a, OC = \cos a, AB = \sin c,$$

$$OA = \cos c, CAB = a.$$

$$\therefore AC = CB \cot a = \sin a \cot a, \quad (a)$$

$$AC = AB \cos a = \sin c \cos a, \quad (b)$$

$$AC = OC \sin b = \cos a \sin b, \quad (c)$$

$$AC = OA \tan b = \cos c \tan b. \quad (d)$$

Equating these values of AC , we obtain the following formulas :

$$\begin{array}{ll} \text{From (a) and (b),} & \sin a = \sin c \sin a. \\ \text{Permuting,} & \sin b = \sin c \sin \beta. \end{array} \quad (5)$$

$$\begin{array}{ll} \text{From (a) and (c),} & \sin b = \tan a \cot a. \\ \text{Permuting,} & \sin a = \tan b \cot \beta. \end{array} \quad (6)$$

$$\text{From (a) and (d),} \quad \cos c = \frac{\sin a}{\tan b} \cot a;$$

$$\therefore \text{ from (6),} \quad \cos c = \cot a \cot \beta. \quad (2)$$

$$\text{From (b) and (c),} \quad \cos a = \cos a \frac{\sin b}{\sin c};$$

$\therefore$ from the sine proportion or from (5)

$$\begin{array}{ll} & \cos a = \cos a \sin \beta. \\ \text{Permuting,} & \cos \beta = \cos b \sin a. \end{array} \quad (3)$$

$$\begin{array}{ll} \text{From (b) and (d),} & \cos a = \tan b \cot c. \\ \text{Permuting,} & \cos \beta = \tan a \cot c. \end{array} \quad (4)$$

$$\text{From (c) and (d),} \quad \cos c = \cos a \cos b. \quad (1)$$

127. Napier's Rules.—Napier, the celebrated Scotch mathematician, devised two rules by which the ten formulas connecting the elements of a right spherical triangle may be easily written.

He called the sides a and b about the right angle, and the complements of the two oblique angles and of the hypotenuse, the *parts* of the triangle, not considering the right angle as a part; the parts, then, are a , b , $90^\circ - c$, $90^\circ - \alpha$, $90^\circ - \beta$, which we shall call a , b , c' , α' , β' . By reference to the circular figure, in which the parts are arranged in their order in going around the triangle, it will be seen that if any three parts are considered, either one will lie between the two others, being adjacent to both, or one will be separated from the other two by intermediate parts. Thus b lies between α' and a , being adjacent to both, and β' is separated from both α' and b . The part which lies between two others adjacent to it or is separated from both the others

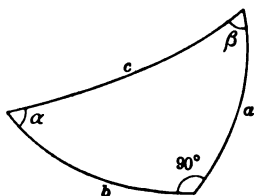


FIG. 105.

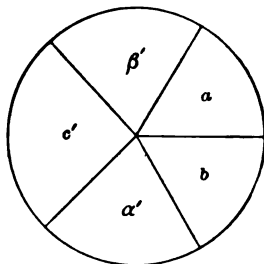


FIG. 106.

by intervening parts, is called the *middle part*; the two others, if adjacent to it, are called *adjacent parts*, and if separated, *opposite parts*. Thus, if c' , α' , and b are considered, α' is the middle part, and c' and b are the adjacent parts; if c' , β' , and b are considered, b is the middle part, and c' and β' are the opposite parts.

Napier's rules are :

1. *The sine of the middle part is equal to the product of the tangents of the adjacent parts.*
2. *The sine of the middle part is equal to the product of the cosines of the opposite parts.*

The rules may be easily remembered by the a in the words *tangent* and *adjacent* and the o in *cosine* and *opposite*.

If, in a right triangle, any two elements besides the right angle are given, the other elements may always be expressed in terms of these two by Napier's rules. Thus, let the given elements be a and c .

(1) To find α ; of the three parts a , c' , and α' , a is the middle part, and c' and α' are the opposite parts.

$$\therefore \sin a = \cos c' \cos a' = \cos (90^\circ - c) \cos (90^\circ - a) = \sin c \sin a.$$

$$\therefore \sin a = \frac{\sin a}{\sin c}.$$

(2) To find β ; of the three parts a , c' , and β' , β' is the middle part, and a and c' are the adjacent parts.

$$\therefore \sin \beta' = \tan a \tan c'.$$

$$\therefore \sin (90^\circ - \beta) = \tan a \tan (90^\circ - c).$$

$$\therefore \cos \beta = \tan a \cot c.$$

(3) To find b ; of the three parts a , c' , and b , c' is the middle part, and a and b are the opposite parts.

$$\therefore \sin c' = \cos a \cos b.$$

$$\therefore \sin (90^\circ - c) = \cos a \cos b.$$

$$\therefore \cos c = \cos a \cos b.$$

$$\therefore \cos b = \frac{\cos c}{\cos a}.$$

128. Species.—Two angular quantities are said to be of the same species when both are less or both greater than 90° , i.e. when they are in the same quadrant; and of different species when they are in different quadrants.

129. Rules for Species in Right Spherical Triangles.

(1) *An oblique angle and its opposite side are always of the same species.* From Napier's rules,

$$\sin b = \tan a \cot \alpha.$$

But $\sin b$ is always positive, and therefore $\tan a$ and $\cot \alpha$ must have the same sign; if they are both positive a and α will be in the first quadrant, and if both are negative a and α will be in the second quadrant.

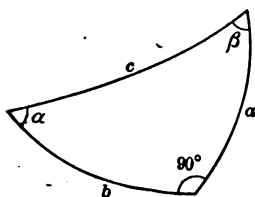


FIG. 107.

(2) *If the hypotenuse is less than 90° , the two oblique angles (and therefore the two sides) are of the same species; if it is greater than 90° , the two angles (and therefore the two sides) are of different species.* From Napier's rules,

$$\cos c = \cot \alpha \cot \beta = \cos a \cos b.$$

If c is less than 90° its cosine will be positive; $\cot \alpha$ and $\cot \beta$ must therefore have the same sign, and hence α and β must be in the same quadrant. If c is greater than 90° its cosine will be negative; $\cot \alpha$ and $\cot \beta$ must therefore have different signs, and hence α and β must be in different quadrants.

Thus, if $\alpha = 40^\circ$ and $\beta = 60^\circ$, α and b must be, from the first rule, in the first quadrant; and, since α and β (or a and b) are in the same quadrant, c must be, from the second rule, in the first quadrant. If $\alpha = 70^\circ$ and $c = 110^\circ$, from the second rule we see that β must be in the second quadrant, and from the first rule that a is in the first and b in the second quadrant.

130. Solution of Right Spherical Triangles.—There are six possible cases, all of which may be solved by Napier's rules :

- I. Given the hypotenuse and an angle.
- II. Given the hypotenuse and a side.
- III. Given the two angles.
- IV. Given the two sides.
- V. Given an angle and the adjacent side.
- VI. Given an angle and the opposite side.

The required elements should always be determined directly from the given elements.

First write the three formulas, each containing the two given elements and one required element; arrange the three formulas for logarithmic computation; and then write the values of the functions in their proper places, being very careful about writing n after the logarithms of the negative functions. If the number of negative factors is even, the result will be positive; if it is odd, the result will be negative and n should be written after the resulting logarithm.

If the sine of a quantity is found by the computation to be positive, the quantity may be either in the first or in the second quadrant, the proper quadrant being determined by the rules for species; if the sine is negative the triangle is impossible, since the elements of the triangle are each less than 180° . If the cosine, tangent, or cotangent is found to be positive, the quantity lies in the first quadrant; if negative, the quantity lies in the second quadrant.

A check formula in each case is found by applying Napier's rules to the three unknown elements; thus, if a and b are given, α , β , and c will be computed, and the check formula is

$$\cos c = \cot \alpha \cot \beta.$$

131. Case I. Given the Hypotenuse and an Angle.

1. $c = 129^\circ 14'.6$, $\alpha = 43^\circ 15'.7$.

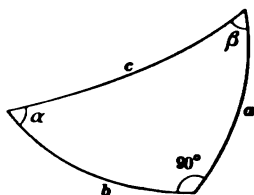


FIG. 105.

To find β ; $\cos c = \cot \alpha \cot \beta$;

$$\therefore \cot \beta = \frac{\cos c}{\cot \alpha}$$

To find b ; $\cos \alpha = \tan b \cot c$;

$$\therefore \tan b = \frac{\cos \alpha}{\cot c}$$

To find a ; $\sin a = \sin \alpha \sin c$.

Check: $\sin a = \tan b \cot \beta$.

$\log \cos c = 9.80114 \pi$	$\log \cos \alpha = 9.86227$	$\log \sin \alpha = 9.83590$
$-\log \cot \alpha = 0.02637$	$-\log \cot c = 9.91214 \pi$	$+\log \sin c = 9.88900$
$\log \cot \beta = 9.77477 \pi$	$\log \tan b = 9.95013 \pi$	$\log \sin a = 9.72490$
$\therefore \beta = 120^\circ 46'.03$	$b = 138^\circ 16'.96$	$a = 32^\circ 3'.4$

By the rules for species β must be in the second, a in the first, and b in the second quadrant.

2. $c = 110^\circ$, $\beta = 48^\circ 28'.6$.

$$\therefore a = 118^\circ 46'.1; b = 44^\circ 42'.7; \alpha = 111^\circ 7'.2.$$

132. Case II. Given the Hypotenuse and a Side.

1. $c = 75^\circ 0'.4$, $a = 32^\circ 56'$. $\therefore b = 72^\circ 2'.8$; $\alpha = 34^\circ 15'.0$; $\beta = 80^\circ 0'.6$.

2. $c = 100^\circ 12'$, $b = 40^\circ 30'.3$. $\therefore a = 103^\circ 28'.1$; $\alpha = 98^\circ 50'.5$; $\beta = 41^\circ 17'.7$.

133. Case III. Given the Two Angles.

1. $\alpha = 30^\circ 51'.2$, $\beta = 71^\circ 36'$. $\therefore a = 25^\circ 12'.8$; $b = 52^\circ 0'.75$; $c = 56^\circ 9'.6$.

2. $\alpha = 130^\circ 20'$, $\beta = 100^\circ 10'.9$. $\therefore a = 131^\circ 7'.0$; $b = 103^\circ 24'.5$; $c = 81^\circ 13'.7$.

134. Case IV. Given the Two Sides.

1. $a = 43^\circ 20'$, $b = 74^\circ 13'$. $\therefore c = 78^\circ 35'.3$; $\alpha = 44^\circ 26'.0$; $\beta = 79^\circ 1'.4$.

2. $a = 100^\circ$, $b = 98^\circ 20'$. $\therefore c = 88^\circ 33'.5$; $\alpha = 99^\circ 53'.8$; $\beta = 98^\circ 12'.5$.

135. Case V. Given One Side and the Adjacent Angle.

1. $b = 66^\circ 29'$, $\alpha = 50^\circ 17'$. $\therefore a = 47^\circ 49'.5$; $c = 74^\circ 27'.6$; $\beta = 72^\circ 7'.5$.

2. $a = 24^\circ 41'$, $\beta = 140^\circ 34'.7$. $\therefore b = 161^\circ 3'.2$; $c = 149^\circ 15'.0$; $\alpha = 54^\circ 46'.6$.

136. Case VI. Given One Side and the Angle Opposite.—

Let a and α be the given elements.

To find b ; $\sin b = \tan a \cot \alpha$.

To find c ; $\sin a = \sin c \sin \alpha$; $\therefore \sin c = \frac{\sin a}{\sin \alpha}$.

To find β ; $\cos a = \cos \alpha \sin \beta$; $\therefore \sin \beta = \frac{\cos a}{\cos \alpha}$.

All three quantities are found by their sines; each of the three quantities may therefore be in either the first or the second quadrant, and there will be two solutions.*

If P is the pole of AC , PC will be perpendicular to AC , and the distances BC will be equal if m is equal to n . Either of the triangles ABC will satisfy the conditions of the problem, being right-angled at C and having $A = \alpha$ and $BC = a$.

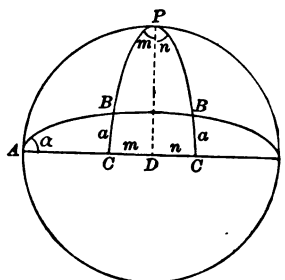


FIG. 109.

The rules for species must be applied in determining which values belong to each solution. Thus, if α is greater than 90° , and β in the first solution is less than 90° , c must be greater than 90° in that triangle; in the second triangle, where β is greater than 90° , c must be less than 90° . Each side, of course, is of the same species as its opposite angle. These results may be written:

First solution: $\alpha > 90^\circ$; $\beta < 90^\circ$; $c > 90^\circ$; $a > 90^\circ$; $b < 90^\circ$.

Second solution: $\alpha > 90^\circ$; $\beta > 90^\circ$; $c < 90^\circ$; $a > 90^\circ$; $b > 90^\circ$.

1. $a = 160^\circ 12'.2$, $\alpha = 150^\circ 37'$.

$\log \tan a = 9.55625 \ n$	$\log \sin a = 9.52979$	$\log \cos a = 9.94020 \ n$
$+ \log \cot \alpha = 0.24942 \ n$	$- \log \sin \alpha = 9.69077$	$- \log \cos \alpha = 9.97354 \ n$
$\log \sin b = 9.80567$	$\log \sin c = 9.83902$	$\log \sin \beta = 9.96666$
$b = 39^\circ 44'.1$	$c' = 43^\circ 39'.1$	$\beta = 67^\circ 50'.2$
$b' = 140^\circ 15'.9$	$c = 136^\circ 20'.9$	$\beta' = 112^\circ 9'.8$

2. $b = 40^\circ 50'$, $\beta = 62^\circ 14'$. $\therefore a = 27^\circ 3'.9$; $a = 38^\circ 0'.4$; $c = 47^\circ 38'.6$;
or $a' = 152^\circ 56'.1$; $a' = 141^\circ 59'.6$; $c' = 132^\circ 21'.4$.

* The triangle is supposed to be possible. The two solutions are identical when $a = \alpha$.

137. Special Cases.

1. $c = 90^\circ$, $\alpha = 90^\circ$. $\therefore a = 90^\circ$; b and β indeterminate.
2. $c = 90^\circ$, $\alpha = 90^\circ$. $\therefore a = 90^\circ$; b and β indeterminate.
3. $\alpha = 90^\circ$, $\beta = 90^\circ$. $\therefore a = 90^\circ$; $b = 90^\circ$; $c = 90^\circ$.
4. $\alpha = 90^\circ$, $b = 90^\circ$. $\therefore c = 90^\circ$; $a = 90^\circ$; $\beta = 90^\circ$.
5. $\alpha = 90^\circ$, $\beta = 90^\circ$. $\therefore c = 90^\circ$; $b = 90^\circ$; $a = 90^\circ$.
6. $a = 20^\circ$, $\alpha = 20^\circ$. $\therefore c = 90^\circ$; $b = 90^\circ$; $\beta = 90^\circ$.

138. Additional Examples.

1. $a = 40^\circ 42'.4$, $c = 63^\circ 20'$.
 $\therefore b = 53^\circ 41'.9$; $\alpha = 46^\circ 52'.25$; $\beta = 64^\circ 24'.0$.
2. $a = 70^\circ 15'.5$, $\alpha = 81^\circ 42'.7$.
 $\therefore b = 23^\circ 57'.0$; $\beta = 25^\circ 15'.7$; $c = 72^\circ 1'.25$;
or $b' = 156^\circ 3'.0$; $\beta' = 154^\circ 44'.3$; $c' = 107^\circ 58'.75$.
3. $b = 30^\circ 32'.4$, $\alpha = 36^\circ 44'$.
 $\therefore a = 20^\circ 46'.0$; $c = 36^\circ 21'.6$; $\beta = 58^\circ 59'.7$.
4. $c = 72^\circ 10'$, $\alpha = 30^\circ 43'$.
 $\therefore a = 29^\circ 5'.6$; $b = 69^\circ 29'.0$; $\beta = 79^\circ 41'.25$.
5. $\alpha = 106^\circ 34'.2$, $\beta = 33^\circ 11'.7$.
 $\therefore a = 121^\circ 23'.6$; $b = 29^\circ 11'.0$; $c = 117^\circ 3'.0$.
6. $a = 28^\circ 47'$, $b = 110^\circ 27'.3$.
 $\therefore c = 107^\circ 50'.2$; $\alpha = 30^\circ 23'.1$; $\beta = 100^\circ 10'.9$.
7. $c = 54^\circ 12'.2$, $\beta = 164^\circ 50'.4$.
 $\therefore a = 99^\circ 0'.3$; $b = 167^\circ 45'.2$; $\alpha = 126^\circ 45'.9$.
8. $a = 40^\circ 8'$, $\beta = 74^\circ 30'.2$.
 $\therefore b = 66^\circ 43'.5$; $c = 72^\circ 25'.0$; $\alpha = 42^\circ 32'.7$.
9. $c = 102^\circ 36'$, $\alpha = 125^\circ 13'.4$.
 $\therefore a = 127^\circ 8'.1$; $b = 68^\circ 49'.0$; $\beta = 72^\circ 49'.8$.
10. $a = 40^\circ 42'.4$, $\beta = 67^\circ 51'.6$.
 $\therefore a = 35^\circ 4'.4$; $b = 54^\circ 42'.0$; $c = 61^\circ 46'.6$.
11. $b = 163^\circ 14'.2$, $c = 112^\circ 41'.8$.
 $\therefore a = 66^\circ 14'.1$; $\alpha = 82^\circ 45'.75$; $\beta = 161^\circ 46'.9$.
12. $a = 120^\circ 30'.2$, $b = 140^\circ 12'$.
 $\therefore c = 67^\circ 2'.8$; $\alpha = 110^\circ 39'.7$; $\beta = 135^\circ 57'.7$.
13. $c = 50^\circ 20'.2$, $\beta = 101^\circ 29'.4$.
 $\therefore a = 166^\circ 29'.5$; $b = 131^\circ 1'.7$; $\alpha = 162^\circ 20'.1$.
14. $\alpha = 82^\circ 4'.4$, $\beta = 8^\circ 22'.3$.
 $\therefore a = 18^\circ 42'.2$; $c = 18^\circ 53'.25$; $b = 2^\circ 43'$ or $2^\circ 44'$.

15. $a = 130^\circ 40'.7$, $c = 75^\circ 31'.5$.

$$\therefore b = 112^\circ 33'.0$$
; $\alpha = 128^\circ 26'.6$; $\beta = 107^\circ 28'.75$.

16. $b = 10^\circ 10'.2$, $\beta = 15^\circ 40'.6$.

$$\therefore a = 39^\circ 43'.9$$
; $c = 40^\circ 48'.1$; $\alpha = 78^\circ 0'.7$;

or $a' = 140^\circ 16'.1$; $c' = 139^\circ 11'.9$; $\alpha' = 101^\circ 59'.3$.

17. $b = 57^\circ 8'.3$, $\alpha = 104^\circ 16'.2$.

$$\therefore a = 106^\circ 50'.8$$
; $c = 99^\circ 2'.8$; $\beta = 58^\circ 16'.4$.

18. $\alpha = 20^\circ 54'$, $b = 64^\circ 26'.7$.

$$\therefore c = 66^\circ 14'.1$$
; $a = 22^\circ 56'.5$; $\beta = 80^\circ 19'.2$.

139. Isosceles Triangles.—If an arc of a great circle be drawn from the vertex perpendicular to the base, it will bisect both the base and the angle at the vertex, dividing the triangle into two equal right triangles that may be solved by Napier's rules.

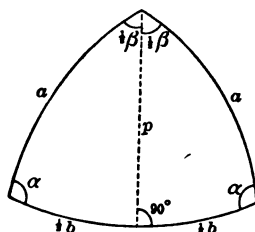


FIG. 110.

1. $a = 110^\circ 47'.3$, $\beta = 92^\circ 14'.6$.

$$\therefore \frac{1}{2}\beta = 46^\circ 7'.3$$
.

To find α : $\cos a = \cot \alpha \cot \frac{1}{2}\beta$;

$$\therefore \cot \alpha = \frac{\cos a}{\cot \frac{1}{2}\beta}$$
.

To find $\frac{1}{2}b$: $\sin \frac{1}{2}b = \sin a \sin \frac{1}{2}\beta$.

To find p : $\cos \frac{1}{2}\beta = \tan p \cot a$; $\therefore \tan p = \frac{\cos \frac{1}{2}\beta}{\cot a}$.

Check: $\sin \frac{1}{2}b = \tan p \cot a$.

$\log \cos a = 9.55013 \, n$	$\log \sin a = 9.97076$	$\log \cos \frac{1}{2}\beta = 9.84081$
$-\log \cot \frac{1}{2}\beta = 9.98299$	$+\log \sin \frac{1}{2}\beta = 9.85783$	$-\log \cot a = 9.57936 \, n$
$\log \cot a = 9.56714 \, n$	$\log \sin \frac{1}{2}b = 9.82859$	$\log \tan p = 0.26145 \, n$
<u>$110^\circ 15'.54$</u>	<u>$\frac{1}{2}b = 42^\circ 22'.1$</u>	<u>$p = 118^\circ 42'.6$</u>
	<u>$b = 84^\circ 44'.2$</u>	

2. $\alpha = 82^\circ 26'$, $\beta = 64^\circ 42'$.

$$\therefore a = 77^\circ 53'.6$$
; $\frac{1}{2}b = 31^\circ 32'.75$.

3. $b = 56^\circ 41'$, $\beta = 112^\circ 44'.6$.

$$\therefore a = 38^\circ 59'.6$$
; $a = 34^\circ 45'.6$;

or $\alpha' = 141^\circ 0'.4$; $\alpha' = 145^\circ 14'.4$.

140. Quadrantal Triangles. — The polar of a quadrantal triangle is a right triangle whose angles are the supplements of the sides, and whose sides are the supplements of the angles, of the original triangle.

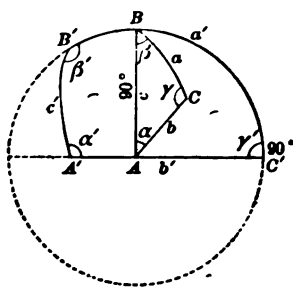


FIG. 111.

We may therefore solve the polar by Napier's rules, and then find the elements of the original triangle by taking the supplements of the elements of the polar.

1. $c = 90^\circ$, $\alpha = 23^\circ 14'.7$, $b = 27^\circ 14'.6$.

$\therefore \gamma' = 90^\circ$, $a' = 156^\circ 45'.3$, $\beta' = 152^\circ 45'.4$

are the elements of the polar triangle.*

To find c' : $\cos \beta' = \tan a' \cot c'$; $\therefore \cot c' = \frac{\cos \beta'}{\tan a'}$.

To find b' : $\sin a' = \tan b' \cot \beta'$; $\therefore \tan b' = \frac{\sin a'}{\cot \beta'}$.

To find a' : $\cos a' = \cos a' \sin \beta'$.

Check: $\cos a' = \cot c' \tan b'$.

$\log \cos \beta' = 9.94894 \text{ n}$	$\log \sin a' = 9.59623$	$\log \cos a' = 9.96324 \text{ n}$
$-\log \tan a' = 9.63300 \text{ n}$	$-\log \cot \beta' = 0.28828 \text{ n}$	$+\log \sin \beta' = 9.66065$
$\log \cot c' = 0.31594$	$\log \tan b' = 9.30795 \text{ n}$	$\log \cos a' = 9.62389 \text{ n}$
$c' = 64^\circ 12'.8$	$b' = 168^\circ 30'.8$	$a' = 114^\circ 52'.4$
$\therefore \gamma = 115^\circ 47'.2$	$\therefore \beta = 11^\circ 29'.2$	$\therefore a = 65^\circ 7'.6$

2. $c = 90^\circ$, $\gamma = 98^\circ 22'.7$, $a = 150^\circ 47'$.

$\therefore a = 150^\circ 26'.2$; $b = 94^\circ 43'.5$; $\beta = 99^\circ 36'.6$.

3. $c = 90^\circ$, $a = 121^\circ 30'$, $\beta = 112^\circ 16'.2$.

$\therefore b = 108^\circ 51'.1$; $\alpha = 123^\circ 30'.75$; $\gamma = 102^\circ 4'.7$.

4. $c = 90^\circ$, $a = 138^\circ 47'.8$, $b = 107^\circ 54'.9$.

$\therefore \alpha = 142^\circ 15'.2$; $\beta = 117^\circ 50'.25$; $\gamma = 111^\circ 40'.1$.

5. $c = 90^\circ$, $a = 112^\circ 6'.5$, $\gamma = 74^\circ 30'$.

$\therefore b = 56^\circ 39'.6$; $\alpha = 116^\circ 46'.4$; $\beta = 53^\circ 36'.9$.

6. $c = 90^\circ$, $a = 83^\circ 20'.6$, $\beta = 77^\circ 14'.3$.

$\therefore \alpha = 83^\circ 30'.3$; $b = 77^\circ 19'.3$; $\gamma = 91^\circ 28'.0$.

7. $c = 90^\circ$, $a = 94^\circ 22'.2$, $\alpha = 108^\circ 13'.3$.

$\therefore b = 14^\circ 6'.2$; $\beta = 13^\circ 25'.3$; $\gamma = 72^\circ 17'.5$;
or $b' = 165^\circ 53'.8$; $\beta' = 166^\circ 34'.7$; $\gamma' = 107^\circ 42'.5$.

* Note that α' , β' , and c' are not the parts of the right triangle, but their complements.

141. Quadrantal Triangles may also be solved by the use of Fig. 112, in which B , one of the vertices adjacent to the quadrantal side, is the pole of the great circle $MDGN$.

If the triangle has one side less than 90° , as BC in the triangle ABC , produce that side to D . In the triangle ACD , $ADC = 90^\circ$, $DAC = 90^\circ - \alpha$, $ACD = 180^\circ - \gamma$, $AC = b$, $CD = 90^\circ - a$, and $AD = \beta$ since $AD = ABD$. Therefore, if any two elements of ACB besides the quadrant are given, we know two elements of the right triangle ACD in addition to the right angle. Hence we could solve it by Napier's rules, thence obtaining the elements of ABC .

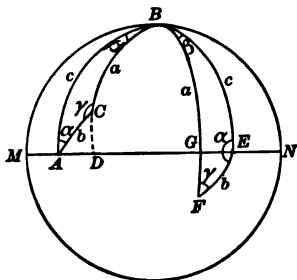


FIG. 112.

If one side of the triangle is greater than 90° , as in BEF , then in the triangle GEF we have $GF = a - 90^\circ$, $GE = \beta$, $EF = b$, $FGE = 90^\circ$, $GEF = \alpha - 90^\circ$, and $GFE = \gamma$. If any two elements of BFE besides the quadrantal side are given, we then know two elements of the triangle GFE in addition to the right angle. Hence we could solve it by Napier's rules, thence finding the elements of BFE .

CHAPTER XI.

OBLIQUE SPHERICAL TRIANGLES.

142. To find an Angle, having given the Three Sides.

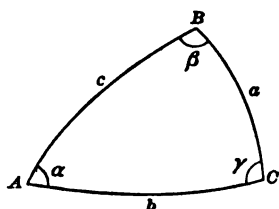


FIG. 113.

$$(a) \quad \cos a = \cos b \cos c + \sin b \sin c \cos \alpha;$$

$$\therefore \cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}, \quad (1)$$

which may be solved by the use of the natural functions.

(b)* To adapt (1) to logarithmic computation, subtract each member from unity.

$$\therefore 1 - \cos \alpha = 1 - \frac{\cos a - \cos b \cos c}{\sin b \sin c} = \frac{\sin b \sin c - \cos a + \cos b \cos c}{\sin b \sin c}$$

$$\therefore 2 \sin^2 \frac{1}{2} \alpha = \frac{\cos(b-c) - \cos a}{\sin b \sin c}.$$

Applying (4) of Art. 73,

$$\cos u - \cos v = -2 \sin \frac{1}{2}(u+v) \sin \frac{1}{2}(u-v),$$

$$\begin{aligned} \text{we have } \cos(b-c) - \cos a &= -2 \sin \frac{1}{2}(b-c+a) \sin \frac{1}{2}(b-c-a) \\ &= +2 \sin \frac{1}{2}(a+b-c) \sin \frac{1}{2}(a-b+c), \end{aligned}$$

since $\sin(-x) = -\sin x$.

$$\begin{aligned} \text{Let } a + b + c &= 2s; \quad \therefore a + b - c = 2s - 2c = 2(s-c); \\ a - b + c &= 2s - 2b = 2(s-b). \end{aligned}$$

$$\therefore \cos(b-c) - \cos a = 2 \sin(s-c) \sin(s-b).$$

$$\therefore \sin^2 \frac{1}{2} \alpha = \frac{\sin(s-b) \sin(s-c)}{\sin b \sin c}.$$

Permuting,

$$\left. \begin{aligned} \sin^2 \frac{1}{2} \beta &= \frac{\sin(s-a) \sin(s-c)}{\sin a \sin c}, \\ \sin^2 \frac{1}{2} \gamma &= \frac{\sin(s-a) \sin(s-b)}{\sin a \sin b}. \end{aligned} \right\} \quad (2)$$

* Compare with Art. 99.

(c)* Add each member of (1) to unity.

$$\begin{aligned}\therefore 1 + \cos a &= 1 + \frac{\cos a - \cos b \cos c}{\sin b \sin c} \\ &= \frac{\cos a - (\cos b \cos c - \sin b \sin c)}{\sin b \sin c} \\ \therefore 2 \cos^2 \frac{1}{2} a &= \frac{\cos a - \cos(b+c)}{\sin b \sin c}.\end{aligned}$$

Applying (4) of Art. 73, we have

$$\begin{aligned}\cos a - \cos(b+c) &= -2 \sin \frac{1}{2}(a+b+c) \sin \frac{1}{2}(a-b-c) \\ &= +2 \sin \frac{1}{2}(a+b+c) \sin \frac{1}{2}(b+c-a).\end{aligned}$$

Let $a+b+c=2s$; $\therefore b+c-a=2s-2a=2(s-a)$.

$$\therefore \cos a - \cos(b+c) = 2 \sin s \sin(s-a).$$

$$\begin{aligned}\therefore \cos^2 \frac{1}{2} a &= \frac{\sin s \sin(s-a)}{\sin b \sin c} \\ \text{Permuting, } \cos^2 \frac{1}{2} \beta &= \frac{\sin s \sin(s-b)}{\sin a \sin c} \\ \cos^2 \frac{1}{2} \gamma &= \frac{\sin s \sin(s-c)}{\sin a \sin b}.\end{aligned} \quad (3)$$

(d) Dividing $\sin^2 \frac{1}{2} a$ by $\cos^2 \frac{1}{2} a$, we have

$$\begin{aligned}\tan^2 \frac{1}{2} a &= \frac{\sin(s-b) \sin(s-c)}{\sin s \sin(s-a)} \\ \text{Permuting, } \tan^2 \frac{1}{2} \beta &= \frac{\sin(s-a) \sin(s-c)}{\sin s \sin(s-b)} \\ \tan^2 \frac{1}{2} \gamma &= \frac{\sin(s-a) \sin(s-b)}{\sin s \sin(s-c)}.\end{aligned} \quad (4)$$

We may write

$$\tan \frac{1}{2} a = \frac{1}{\sin(s-a)} \sqrt{\frac{\sin(s-a) \sin(s-b) \sin(s-c)}{\sin s}}.$$

$$\text{Let } r = \sqrt{\frac{\sin(s-a) \sin(s-b) \sin(s-c)}{\sin s}}. \quad (5)$$

$$\begin{aligned}\therefore \tan \frac{1}{2} a &= \frac{r}{\sin(s-a)} \\ \text{Permuting, } \tan \frac{1}{2} \beta &= \frac{r}{\sin(s-b)} \\ \tan \frac{1}{2} \gamma &= \frac{r}{\sin(s-c)}.\end{aligned} \quad (6)$$

* Compare with Art. 99.

NOTE.—The center of the inscribed circle of a spherical triangle is the point of intersection of the arcs of great circles bisecting the angles of the triangle. From this point O draw the arcs OL , OM , and ON perpendicular to the sides of the triangle.

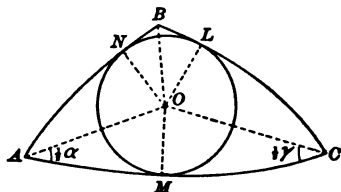


FIG. 114.

$$\therefore MA = AN; NB = BL; CM = LC.$$

$$\therefore MA + NB + CM = s.$$

$$\therefore b + NB = s, \text{ since } MA + CM = b.$$

$$\therefore NB = s - b.$$

In the right triangle ONB , by Napier's rules,

$$\sin NB = \tan ON \cot NBO.$$

$$\begin{aligned} \therefore \tan ON &= \frac{\sin NB}{\cot NBO} = \sin(s-b) \tan \frac{1}{2}\beta \\ &= \sin(s-b) \sqrt{\frac{\sin(s-a) \sin(s-c)}{\sin s \sin(s-b)}} \\ &= \sqrt{\frac{\sin(s-a) \sin(s-b) \sin(s-c)}{\sin s}} \end{aligned}$$

$$\therefore \tan ON = r.$$

Hence r is the tangent of the radius of the inscribed circle.

143. To find a Side, having given the Three Angles.

$$(a) \quad \cos \alpha = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a.$$

$$\therefore \cos a = \frac{\cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma}, \quad (1)$$

which may be solved by the use of the natural functions.

(b)* To adapt (1) to logarithmic computation, subtract each member from unity.

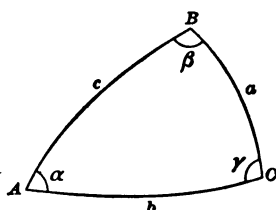


FIG. 115.

$$\begin{aligned} \therefore 1 - \cos a &= 1 - \frac{\cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma} \\ &= -\frac{\cos \beta \cos \gamma - \sin \beta \sin \gamma + \cos \alpha}{\sin \beta \sin \gamma} \end{aligned}$$

$$\therefore 2 \sin^2 \frac{1}{2}a = -\frac{\cos(\beta + \gamma) + \cos \alpha}{\sin \beta \sin \gamma}.$$

Applying the equation (Art. 73)

$$\cos u + \cos v = 2 \cos \frac{1}{2}(u + v) \cos \frac{1}{2}(u - v),$$

we have $\cos(\beta + \gamma) + \cos \alpha = 2 \cos \frac{1}{2}(\alpha + \beta + \gamma) \cos \frac{1}{2}(\beta + \gamma - \alpha)$

* Compare with Art. 99.

Let $\alpha + \beta + \gamma = 2S$; $\therefore \beta + \gamma - \alpha = 2S - 2\alpha = 2(S - \alpha)$.

$$\therefore \cos(\beta + \gamma) + \cos \alpha = 2 \cos S \cos(S - \alpha).$$

$$\text{Permuting, } \left. \begin{aligned} \therefore \sin^2 \frac{1}{2} \alpha &= \frac{-\cos S \cos(S - \alpha)}{\sin \beta \sin \gamma} \\ \sin^2 \frac{1}{2} b &= \frac{-\cos S \cos(S - \beta)}{\sin \alpha \sin \gamma} \\ \sin^2 \frac{1}{2} c &= \frac{-\cos S \cos(S - \gamma)}{\sin \alpha \sin \beta} \end{aligned} \right\} \quad (2)$$

(c)* Add each member of (1) to unity.

$$\begin{aligned} \therefore 1 + \cos \alpha &= 1 + \frac{\cos \alpha + \cos \beta \cos \gamma}{\sin \beta \sin \gamma} \\ &= \frac{\cos \alpha + \cos \beta \cos \gamma + \sin \beta \sin \gamma}{\sin \beta \sin \gamma} \end{aligned}$$

$$\therefore 2 \cos^2 \frac{1}{2} \alpha = \frac{\cos \alpha + \cos(\beta - \gamma)}{\sin \beta \sin \gamma}.$$

Applying the equation (Art. 73)

$$\cos u + \cos v = 2 \cos \frac{1}{2}(u + v) \cos \frac{1}{2}(u - v),$$

we have $\cos \alpha + \cos(\beta - \gamma) = 2 \cos \frac{1}{2}(\alpha + \beta - \gamma) \cos \frac{1}{2}(\alpha - \beta + \gamma)$.

Let $\alpha + \beta + \gamma = 2S$; $\therefore \alpha + \beta - \gamma = 2S - 2\gamma = 2(S - \gamma)$;

$$\alpha - \beta + \gamma = 2S - 2\beta = 2(S - \beta).$$

$$\therefore \cos \alpha + \cos(\beta - \gamma) = 2 \cos(S - \beta) \cos(S - \gamma).$$

$$\text{Permuting, } \left. \begin{aligned} \therefore \cos^2 \frac{1}{2} \alpha &= \frac{\cos(S - \beta) \cos(S - \gamma)}{\sin \beta \sin \gamma} \\ \cos^2 \frac{1}{2} b &= \frac{\cos(S - \alpha) \cos(S - \gamma)}{\sin \alpha \sin \gamma} \\ \cos^2 \frac{1}{2} c &= \frac{\cos(S - \alpha) \cos(S - \beta)}{\sin \alpha \sin \beta} \end{aligned} \right\} \quad (3)$$

(d) Dividing $\sin^2 \frac{1}{2} a$ by $\cos^2 \frac{1}{2} a$, we have

$$\text{Permuting, } \left. \begin{aligned} \tan^2 \frac{1}{2} \alpha &= \frac{-\cos S \cos(S - \alpha)}{\cos(S - \beta) \cos(S - \gamma)} \\ \tan^2 \frac{1}{2} b &= \frac{-\cos S \cos(S - \beta)}{\cos(S - \alpha) \cos(S - \gamma)} \\ \tan^2 \frac{1}{2} c &= \frac{-\cos S \cos(S - \gamma)}{\cos(S - \alpha) \cos(S - \beta)} \end{aligned} \right\} \quad (4)$$

* Compare with Art. 99.

We may write

$$\tan \frac{1}{2} a = \cos (S - \alpha) \sqrt{\frac{-\cos S}{\cos (S - \alpha) \cos (S - \beta) \cos (S - \gamma)}}$$

Let
$$R = \sqrt{\frac{-\cos S}{\cos (S - \alpha) \cos (S - \beta) \cos (S - \gamma)}} \quad (5)$$

$$\left. \begin{aligned} \therefore \tan \frac{1}{2} a &= R \cos (S - \alpha), \\ \text{Permuting, } \tan \frac{1}{2} b &= R \cos (S - \beta), \\ \tan \frac{1}{2} c &= R \cos (S - \gamma). \end{aligned} \right\} \quad (6)$$

NOTE. — Since the sum of the angles $2S$ must be between 180° and 540° , S must be between 90° and 270° , so that $\cos S$ is always negative and hence $-\cos S$ is always positive.

NOTE. — The center of the circumscribed circle of a spherical triangle is the point of intersection of the arcs of great circles perpendicular to the sides of the triangle at their middle points.

$$\therefore AN = NB; BL = LC; CM = MA.$$

$$\therefore OAM = OCM; OAN = OBN; OCL = OBL.$$

$$\therefore OAM + OAN + OCL = S,$$

$$\therefore OCL = S - (OAM + OAN) = S - \alpha.$$

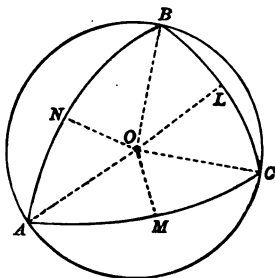


FIG. 116.

In the right triangle OCL , by Napier's rules,

$$\cos OCL = \tan LC \cot OC.$$

$$\begin{aligned} \therefore \tan OC &= \frac{\tan LC}{\cos OCL} = \frac{\tan \frac{1}{2} a}{\cos (S - \alpha)} \\ &= \frac{1}{\cos (S - \alpha)} \sqrt{\frac{-\cos S \cos (S - \alpha)}{\cos (S - \beta) \cos (S - \gamma)}} \\ &= \sqrt{\frac{-\cos S}{\cos (S - \alpha) \cos (S - \beta) \cos (S - \gamma)}} \end{aligned}$$

$$\therefore \tan OC = R.$$

Hence R is the tangent of the radius of the circumscribed circle.

144. Napier's Analogies.

(1) From (4) or (6), Art. 142,

$$\frac{\tan \frac{1}{2} a}{\tan \frac{1}{2} \beta} = \frac{\sin (s - b)}{\sin (s - a)}.$$

By division and composition,

$$\frac{\tan \frac{1}{2} a - \tan \frac{1}{2} \beta}{\tan \frac{1}{2} a + \tan \frac{1}{2} \beta} = \frac{\sin (s - b) - \sin (s - a)}{\sin (s - b) + \sin (s - a)} \quad (a)$$

But

$$\frac{\tan \frac{1}{2} \alpha - \tan \frac{1}{2} \beta}{\tan \frac{1}{2} \alpha + \tan \frac{1}{2} \beta} = \frac{\sin \frac{1}{2} \alpha \cos \frac{1}{2} \beta - \cos \frac{1}{2} \alpha \sin \frac{1}{2} \beta}{\sin \frac{1}{2} \alpha \cos \frac{1}{2} \beta + \cos \frac{1}{2} \alpha \sin \frac{1}{2} \beta} = \frac{\sin \frac{1}{2} (\alpha - \beta)}{\sin \frac{1}{2} (\alpha + \beta)}.$$

Also, from (1) and (2), Art. 73,

$$\begin{aligned} \frac{\sin (s-b) - \sin (s-a)}{\sin (s-b) + \sin (s-a)} &= \frac{2 \cos \frac{1}{2} (2s-a-b) \sin \frac{1}{2} (a-b)}{2 \sin \frac{1}{2} (2s-a-b) \cos \frac{1}{2} (a-b)} \\ &= \frac{\tan \frac{1}{2} (a-b)}{\tan \frac{1}{2} c}. \end{aligned}$$

Substituting these values in (a), and reversing the order,

$$\frac{\tan \frac{1}{2} (a-b)}{\tan \frac{1}{2} c} = \frac{\sin \frac{1}{2} (\alpha - \beta)}{\sin \frac{1}{2} (\alpha + \beta)}. \quad (1)$$

(2) Substituting the values of a, b, c, α , and β in terms of the elements of the polar triangle, (1) becomes

$$\begin{aligned} \frac{\tan \frac{1}{2} (180^\circ - a' - 180^\circ + \beta')}{\tan \frac{1}{2} (180^\circ - \gamma')} &= \frac{\sin \frac{1}{2} (180^\circ - a' - 180^\circ + b')}{\sin \frac{1}{2} (180^\circ - a' + 180^\circ - b')} \\ \therefore \frac{\tan \frac{1}{2} (\beta' - a')}{\cot \frac{1}{2} \gamma'} &= \frac{\sin \frac{1}{2} (b' - a')}{\sin \frac{1}{2} (a' + b')} \\ \therefore \frac{-\tan \frac{1}{2} (a' - \beta')}{\cot \frac{1}{2} \gamma'} &= \frac{-\sin \frac{1}{2} (a' - b')}{\sin \frac{1}{2} (a' + b')}. \end{aligned}$$

Changing the signs and dropping the primes,

$$\frac{\tan \frac{1}{2} (\alpha - \beta)}{\cot \frac{1}{2} \gamma} = \frac{\sin \frac{1}{2} (\alpha - b)}{\sin \frac{1}{2} (\alpha + b)}. \quad (2)$$

(3) From (4), Art. 142,

$$\tan \frac{1}{2} \alpha \tan \frac{1}{2} \beta = \frac{\sin (s-c)}{\sin s}$$

$$\therefore \frac{1 + \tan \frac{1}{2} \alpha \tan \frac{1}{2} \beta}{1 - \tan \frac{1}{2} \alpha \tan \frac{1}{2} \beta} = \frac{\sin s + \sin (s-c)}{\sin s - \sin (s-c)} \quad (b)$$

$$\begin{aligned} \text{But } \frac{1 + \tan \frac{1}{2} \alpha \tan \frac{1}{2} \beta}{1 - \tan \frac{1}{2} \alpha \tan \frac{1}{2} \beta} &= \frac{\cos \frac{1}{2} \alpha \cos \frac{1}{2} \beta + \sin \frac{1}{2} \alpha \sin \frac{1}{2} \beta}{\cos \frac{1}{2} \alpha \cos \frac{1}{2} \beta - \sin \frac{1}{2} \alpha \sin \frac{1}{2} \beta} \\ &= \frac{\cos \frac{1}{2} (\alpha - \beta)}{\cos \frac{1}{2} (\alpha + \beta)}. \end{aligned}$$

Also, from (1) and (2), Art. 73,

$$\frac{\sin s + \sin (s-c)}{\sin s - \sin (s-c)} = \frac{2 \sin \frac{1}{2} (2s-c) \cos \frac{1}{2} c}{2 \cos \frac{1}{2} (2s-c) \sin \frac{1}{2} c} = \frac{\tan \frac{1}{2} (a+b)}{\tan \frac{1}{2} c}.$$

Substituting these values in (b), and reversing the order,

$$\frac{\tan \frac{1}{2}(\alpha + \beta)}{\tan \frac{1}{2}c} = \frac{\cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta)}. \quad (3)$$

(4) Passing to the polar triangle, (3) becomes

$$\frac{\tan \frac{1}{2}(180^\circ - \alpha' + 180^\circ - \beta')}{\tan \frac{1}{2}(180^\circ - \gamma')} = \frac{\cos \frac{1}{2}(180^\circ - \alpha' - 180^\circ + \beta')}{\cos \frac{1}{2}(180^\circ - \alpha' + 180^\circ - \beta')}$$

$$\therefore \frac{\tan [180^\circ - \frac{1}{2}(\alpha' + \beta')]}{\tan (90^\circ - \frac{1}{2}\gamma')} = \frac{\cos \frac{1}{2}(\beta' - \alpha')}{\cos [180^\circ - \frac{1}{2}(\alpha' + \beta')]}.$$

$$\therefore \frac{-\tan \frac{1}{2}(\alpha' + \beta')}{\cot \frac{1}{2}\gamma'} = \frac{\cos \frac{1}{2}(\alpha' - \beta')}{-\cos \frac{1}{2}(\alpha' + \beta')}.$$

Changing the signs and dropping the primes,

$$\frac{\tan \frac{1}{2}(\alpha + \beta)}{\cot \frac{1}{2}\gamma} = \frac{\cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}(\alpha + \beta)}. \quad (4)$$

Eqs. (1), (2), (3), and (4) are called *Napier's Analogies*.

145. **Gauss's Equations.**—From (2) and (3), Art. 142, we have

$$\sin \frac{1}{2}\alpha \cos \frac{1}{2}\beta = \frac{\sin(s-b)}{\sin c} \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}} = \frac{\sin(s-b)}{\sin c} \cos \frac{1}{2}\gamma;$$

$$\cos \frac{1}{2}\alpha \sin \frac{1}{2}\beta = \frac{\sin(s-a)}{\sin c} \sqrt{\frac{\sin s \sin(s-c)}{\sin a \sin b}} = \frac{\sin(s-a)}{\sin c} \cos \frac{1}{2}\gamma;$$

$$\cos \frac{1}{2}\alpha \cos \frac{1}{2}\beta = \frac{\sin s}{\sin c} \sqrt{\frac{\sin(s-a) \sin(s-b)}{\sin a \sin b}} = \frac{\sin s}{\sin c} \sin \frac{1}{2}\gamma;$$

$$\begin{aligned} \sin \frac{1}{2}\alpha \sin \frac{1}{2}\beta &= \frac{\sin(s-c)}{\sin c} \sqrt{\frac{\sin(s-a) \sin(s-b)}{\sin a \sin b}} \\ &= \frac{\sin(s-c)}{\sin c} \sin \frac{1}{2}\gamma. \end{aligned}$$

$$\begin{aligned} (1) \quad \sin \frac{1}{2}(\alpha + \beta) &= \frac{\cos \frac{1}{2}\gamma}{\sin c} [\sin(s-b) + \sin(s-a)] \\ &= \frac{\cos \frac{1}{2}\gamma \sin[s - \frac{1}{2}(\alpha + \beta)] \cos \frac{1}{2}(\alpha - \beta)}{\sin \frac{1}{2}c \cos \frac{1}{2}c} \\ &= \frac{\cos \frac{1}{2}\gamma \cos \frac{1}{2}(\alpha - \beta)}{\cos \frac{1}{2}c}. \end{aligned}$$

$$\therefore \cos \frac{1}{2}c \sin \frac{1}{2}(\alpha + \beta) = \cos \frac{1}{2}\gamma \cos \frac{1}{2}(\alpha - \beta). \quad (1)$$

$$\begin{aligned}
 (2) \quad \cos \frac{1}{2}(\alpha + \beta) &= \frac{\sin \frac{1}{2}\gamma}{\sin c} [\sin s - \sin(s - c)] \\
 &= \frac{\sin \frac{1}{2}\gamma \cos(s - \frac{1}{2}c) \sin \frac{1}{2}c}{\sin \frac{1}{2}c \cos \frac{1}{2}c} \\
 &= \frac{\sin \frac{1}{2}\gamma \cos \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}c}. \\
 \therefore \cos \frac{1}{2}c \cos \frac{1}{2}(\alpha + \beta) &= \sin \frac{1}{2}\gamma \cos \frac{1}{2}(\alpha + \beta). \quad (2)
 \end{aligned}$$

$$\begin{aligned}
 (3) \quad \sin \frac{1}{2}(\alpha - \beta) &= \frac{\cos \frac{1}{2}\gamma}{\sin c} [\sin(s - b) - \sin(s - a)] \\
 &= \frac{\cos \frac{1}{2}\gamma \cos[s - \frac{1}{2}(\alpha + b)] \sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}c \cos \frac{1}{2}c} \\
 &= \frac{\cos \frac{1}{2}\gamma \sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}c}. \\
 \therefore \sin \frac{1}{2}c \sin \frac{1}{2}(\alpha - \beta) &= \cos \frac{1}{2}\gamma \sin \frac{1}{2}(a - b). \quad (3)
 \end{aligned}$$

$$\begin{aligned}
 (4) \quad \cos \frac{1}{2}(\alpha - \beta) &= \frac{\sin \frac{1}{2}\gamma}{\sin c} [\sin s + \sin(s - c)] \\
 &= \frac{\sin \frac{1}{2}\gamma \sin(s - \frac{1}{2}c) \cos \frac{1}{2}c}{\sin \frac{1}{2}c \cos \frac{1}{2}c} \\
 &= \frac{\sin \frac{1}{2}\gamma \sin \frac{1}{2}(\alpha + b)}{\sin \frac{1}{2}c}. \\
 \therefore \sin \frac{1}{2}c \cos \frac{1}{2}(\alpha - \beta) &= \sin \frac{1}{2}\gamma \sin \frac{1}{2}(\alpha + b). \quad (4)
 \end{aligned}$$

Eqs. (1), (2), (3), and (4) are known as *Gauss's Equations*, or *Delambre's Analogies*.

146. Rules for Species in Oblique Spherical Triangles.

(1) *If a side (or angle) differs more than another side (or angle) from 90°, it is of the same species as its opposite angle (or side).*

We wish to show that $\cos a$ and $\cos \alpha$ will have the same sign when the difference between a and 90° is numerically greater than the difference between b and 90° . In the formula

$$\cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \quad (1)$$

the denominator is always positive, so that the sign of the fraction, and hence that of $\cos \alpha$, is the same as that of the numera-

1. Solve the triangle when
- $a = 114^\circ 43'.3$
- ,
- $b = 136^\circ 19'.6$
- ,
- $c = 43^\circ 18'.5$
- .

$s = 147^\circ 10'.7$	$\text{col sin } s = 0.26598$	$\therefore \log \tan \frac{1}{2}a = 9.89910$
$s - a = 32^\circ 27'.4$	$\log \sin (s - a) = 9.72970$	$\frac{1}{2}a = 38^\circ 24'.2$
$s - b = 10^\circ 51'.1$	$\log \sin (s - b) = 9.27478$	$\log \tan \frac{1}{2}\beta = 0.35402$
$s - c = 103^\circ 52'.2$	$\log \sin (s - c) = 9.98714$	$\frac{1}{2}\beta = 66^\circ 7'.6$
$2s = 294^\circ 21'.4$	$\log r^2 = 29.25760 - 30$	$\log \tan \frac{1}{2}\gamma = 9.64166$
$a \text{ check.}$	$\log r = 9.62880$	$\frac{1}{2}\gamma = 23^\circ 39'.7$

In finding $\log \tan \frac{1}{2}a$ write $\log r$ on the margin of a slip of paper, place it above $\log \sin (s - a)$, and write the difference opposite $\log \tan \frac{1}{2}a$; then find $\log \tan \frac{1}{2}\beta$ and $\log \tan \frac{1}{2}\gamma$ in the same manner.

- 2.
- $a = 76^\circ 40'.4$
- ,
- $b = 54^\circ 21'.3$
- ,
- $c = 36^\circ 8'.7$
- .

$$\therefore \frac{1}{2}a = 60^\circ 1'.8; \quad \frac{1}{2}\beta = 23^\circ 8'.6; \quad \frac{1}{2}\gamma = 15^\circ 49'.3.$$

- 3.
- $a = 124^\circ 34'.9$
- ,
- $b = 66^\circ 7'.2$
- ,
- $c = 109^\circ 43'.5$
- .

$$\therefore \frac{1}{2}a = 60^\circ 1'.3; \quad \frac{1}{2}\beta = 37^\circ 0'.8; \quad \frac{1}{2}\gamma = 49^\circ 6'.8.$$

- 4.
- $a = 30^\circ 17'.6$
- ,
- $b = 22^\circ 14'.4$
- ,
- $c = 18^\circ 51'.8$
- .

$$\therefore \frac{1}{2}a = 47^\circ 55'.0; \quad \frac{1}{2}\beta = 24^\circ 8'.5; \quad \frac{1}{2}\gamma = 19^\circ 48'.45.$$

- 5.
- $a = 130^\circ 46'.0$
- ,
- $b = 113^\circ 21'.4$
- ,
- $c = 102^\circ 16'.2$
- .

$$\therefore \frac{1}{2}a = 72^\circ 38'.0; \quad \frac{1}{2}\beta = 68^\circ 9'.6; \quad \frac{1}{2}\gamma = 66^\circ 20'.5.$$

149. Case II. Given the Three Angles (α , β , γ). — Find the sides by the formulas

$$R = \sqrt{\frac{-\cos S}{\cos (S - \alpha) \cos (S - \beta) \cos (S - \gamma)}}, \quad \left. \begin{array}{l} \tan \frac{1}{2}a = R \cos (S - \alpha). \\ \tan \frac{1}{2}b = R \cos (S - \beta). \\ \tan \frac{1}{2}c = R \cos (S - \gamma). \end{array} \right\}$$

Check by the sine proportion.

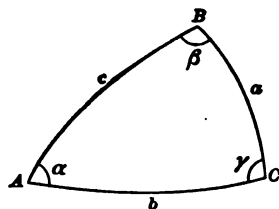


FIG. 118.

1. Solve the triangle when
- $a = 116^\circ 19'.4$
- ,
- $\beta = 83^\circ 19'.2$
- ,
- $\gamma = 106^\circ 10'.6$
- .

$S = 152^\circ 54'.6$	$\text{col } (-\cos S) = 0.05047$	$\therefore \log \tan \frac{1}{2}a = 0.23789$
$S - a = 36^\circ 35'.2$	$\log \cos (S - a) = 9.90469$	$\frac{1}{2}a = 59^\circ 57'.7$
$S - \beta = 69^\circ 35'.4$	$\log \cos (S - \beta) = 9.54249$	$\log \tan \frac{1}{2}b = 9.87569$
$S - \gamma = 46^\circ 44'.0$	$\log \cos (S - \gamma) = 9.83594$	$\frac{1}{2}b = 36^\circ 54'.6$
$2S = 305^\circ 49'.2$	$\text{col } R^2 = 9.33359$	$\log \tan \frac{1}{2}c = 0.16914$
$a \text{ check.}$	$\log R^2 = 0.66641$	$\frac{1}{2}c = 55^\circ 53'.1$
	$\log R = 0.33320$	

In finding $\log \tan \frac{1}{2} a$, write $\log R$ on the margin of a slip of paper, place it above $\log \cos (\beta - \alpha)$, and write the sum opposite $\log \tan \frac{1}{2} a$; then find $\log \tan \frac{1}{2} b$ and $\log \tan \frac{1}{2} c$ in a similar manner.

2. $\alpha = 110^\circ 38'.4$, $\beta = 122^\circ 8'.7$, $\gamma = 140^\circ 20'.3$.
 $\therefore \frac{1}{2} a = 41^\circ 56'.3$; $\frac{1}{2} b = 57^\circ 57'.5$; $\frac{1}{2} c = 68^\circ 39'.4$.
3. $\alpha = 120^\circ 50'.6$, $\beta = 78^\circ 6'.1$, $\gamma = 81^\circ 12'.3$.
 $\therefore \frac{1}{2} a = 59^\circ 55'.2$; $\frac{1}{2} b = 40^\circ 40'.1$; $\frac{1}{2} c = 43^\circ 23'.4$.
4. $\alpha = 80^\circ 20'.2$, $\beta = 73^\circ 46'.7$, $\gamma = 54^\circ 8'.5$.
 $\therefore \frac{1}{2} a = 32^\circ 23'.6$; $\frac{1}{2} b = 30^\circ 53'.7$; $\frac{1}{2} c = 24^\circ 1'.7$.
5. $\alpha = 100^\circ 51'.3$, $\beta = 80^\circ 47'.6$, $\gamma = 74^\circ 3'.3$.
 $\therefore \frac{1}{2} a = 49^\circ 22'.4$; $\frac{1}{2} b = 41^\circ 42'.5$; $\frac{1}{2} c = 37^\circ 41'.6$.

150. Case III. Given Two Sides and the Included Angle (b, c, α).—By permuting (6) and (7), Art. 147, we have

$$\tan \frac{1}{2} (\beta - \gamma) = \cot \frac{1}{2} \alpha \frac{\sin \frac{1}{2} (b - c)}{\sin \frac{1}{2} (b + c)}, \quad (1)$$

$$\tan \frac{1}{2} (\beta + \gamma) = \cot \frac{1}{2} \alpha \frac{\cos \frac{1}{2} (b - c)}{\cos \frac{1}{2} (b + c)}. \quad (2)$$

Then

$$\beta = \frac{1}{2} (\beta + \gamma) + \frac{1}{2} (\beta - \gamma),$$

$$\gamma = \frac{1}{2} (\beta + \gamma) - \frac{1}{2} (\beta - \gamma).$$

Note that the larger angle must be opposite the larger side.

To obtain a , we permute (4) and (5), Art. 147 :

$$\tan \frac{1}{2} a = \tan \frac{1}{2} (b - c) \frac{\sin \frac{1}{2} (\beta + \gamma)}{\sin \frac{1}{2} (\beta - \gamma)}, \quad (3)$$

$$\tan \frac{1}{2} a = \tan \frac{1}{2} (b + c) \frac{\cos \frac{1}{2} (\beta + \gamma)}{\cos \frac{1}{2} (\beta - \gamma)}. \quad (4)$$

The agreement of the values of $\frac{1}{2} a$ found from (3) and (4) is a check upon the computation. The sine proportion may also be used as a check.

NOTE.—In using these formulas, the larger side and the larger angle should be written first in the expressions $b - c$ and $\beta - \gamma$. Thus for $c > b$, (1) would be written

$$\tan \frac{1}{2} (\gamma - \beta) = \cot \frac{1}{2} \alpha \frac{\sin \frac{1}{2} (c - b)}{\sin \frac{1}{2} (c + b)}.$$

Eq. (1) may be read: "The tangent of half the difference of the required angles is equal to the cotangent of half the given angle, multiplied by the sine of half the difference of the given sides, and divided by the sine of half their sum."

1. Solve the triangle when $b = 105^\circ 14'.8$, $c = 43^\circ 17'.2$, $\alpha = 112^\circ 47'.4$.

$b = 105^\circ 14'.8$	$\log \cot \frac{1}{2} \alpha = 9.82251$	$\log \cot \frac{1}{2} \alpha = 9.82251$
$c = 43^\circ 17'.2$	$\log \sin \frac{1}{2} (b-c) = 9.71159$	$\log \cos \frac{1}{2} (b-c) = 9.93316$
$\frac{1}{2} (b+c) = 74^\circ 16'.0$	$\text{col sin } \frac{1}{2} (b+c) = 0.01658$	$\text{col cos } \frac{1}{2} (b+c) = 0.56677$
$\frac{1}{2} (b-c) = 30^\circ 58'.8$	$\log \tan \frac{1}{2} (\beta-\gamma) = 9.55068$	$\log \tan \frac{1}{2} (\beta+\gamma) = 0.32244$
$\frac{1}{2} \alpha = 56^\circ 23'.7$	$\frac{1}{2} (\beta-\gamma) = 19^\circ 33'.8$	$\frac{1}{2} (\beta+\gamma) = 64^\circ 32'.9$
		$\frac{1}{2} (\beta-\gamma) = 19^\circ 33'.8$
$\log \tan \frac{1}{2} (b-c) = 9.77843$	$\log \tan \frac{1}{2} (b+c) = 0.55019$	$\beta = 84^\circ 6'.7$
$\log \sin \frac{1}{2} (\beta+\gamma) = 9.95566^*$	$\log \cos \frac{1}{2} (\beta+\gamma) = 9.63322^*$	$\gamma = 44^\circ 59'.1$
$\text{col sin } \frac{1}{2} (\beta-\gamma) = 0.47514^*$	$\text{col cos } \frac{1}{2} (\beta-\gamma) = 0.02582^*$	
$\log \tan \frac{1}{2} \alpha = 0.20923$	$\log \tan \frac{1}{2} \alpha = 0.20923$	
$\frac{1}{2} \alpha = 58^\circ 17'.8$	$\frac{1}{2} \alpha = 58^\circ 17'.8$	

2. $a = 103^\circ 44'.7$, $b = 64^\circ 12'.3$, $\gamma = 98^\circ 33'.8$.

$$\therefore \frac{1}{2} (\alpha + \beta) = 82^\circ 37'.0; \quad \frac{1}{2} (\alpha - \beta) = 16^\circ 19'.0; \quad \alpha = 98^\circ 56'.0;$$

$$\beta = 66^\circ 18'.0; \quad \frac{1}{2} c = 51^\circ 45'.3.$$

3. $a = 156^\circ 12'.2$, $b = 112^\circ 48'.6$, $\gamma = 76^\circ 32'.4$.

$$\therefore \frac{1}{2} (\alpha + \beta) = 120^\circ 45'.6; \quad \frac{1}{2} (\alpha - \beta) = 33^\circ 18'.5; \quad \alpha = 154^\circ 4'.1;$$

$$\beta = 87^\circ 27'.1; \quad \frac{1}{2} c = 31^\circ 54'.4.$$

151. Case III. Second Method. Given b , c , α , to find One Element only.

(1) To find a only.

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha.$$

Let

$$\left. \begin{aligned} m \sin M &= \sin c \cos \alpha, \\ m \cos M &= \cos c. \end{aligned} \right\} \quad (1)$$

$$\therefore \cos a = m (\cos b \cos M + \sin b \sin M).$$

$$\therefore \cos a = m \cos (b - M). \quad (2)$$

(2) To find one angle only, β or γ . — From (6), Art. 124,

$$\sin \alpha \cot \gamma = \cot c \sin b - \cos b \cos \alpha.$$

$$\therefore \cot \gamma = \frac{\cot c \sin b - \cos b \cos \alpha}{\sin \alpha} = \frac{\cos c \sin b - \cos b \sin c \cos \alpha}{\sin \alpha \sin c}.$$

* The functions of $\frac{1}{2} (\beta - \gamma)$ and of $\frac{1}{2} (\beta + \gamma)$ should be found by using the fraction from which the decimal of a minute is found. Thus,

$$\log \sin \frac{1}{2} (\beta + \gamma) = 9.95561 + \frac{2}{3} \times 6 = 9.95561 + 5 = 9.95566.$$

$$\text{Let} \quad \left. \begin{aligned} m \sin M &= \sin c \cos a, \\ m \cos M &= \cos c. \end{aligned} \right\} \quad (3)$$

$$\therefore \cot \gamma = \frac{m \sin (b - M)}{\sin a \sin c}. \quad (4)$$

The formula for $\cot \beta$ may be found by permuting b and c in (3) and (4).

$$1. \quad b = 105^\circ 14'.8, \quad c = 43^\circ 17'.2, \quad a = 112^\circ 47'.4.$$

To find a .		To find γ .		To find β .	
$\log \sin c = 9.83611$	(1)	$\log \sin c =$	(1)	$\log \sin b = 9.98444$	
$\log \cos a = 9.58811 \text{ } n$	(3)	$\log \cos a =$	(3)	$\log \cos a = 9.58811 \text{ } n$	
$\log(m \sin M)$		$\log(m \sin M)$		$\log(m \sin M)$	
$= 9.42422 \text{ } n$	(4)	$=$	(5)	$= 9.57255 \text{ } n$	
$\log \sin M = 9.53499 \text{ } n$	(7)	$\log \sin M =$	(8)	$\log \sin M = 9.91266 \text{ } n$	
$\log \cos M = 9.97286$	(8)	$\log \cos M =$	(9)	$\log \cos M = 9.78002 \text{ } n$	
$\log \cos c = 9.86209$	(2)	$\log \cos c =$	(2)	$\log \cos b = 9.41992 \text{ } n$	
$\log \tan M = 9.56213 \text{ } n$	(5)	$\log \tan M = 9.56213 \text{ } n$	(6)	$\log \tan M = 0.15263$	
$M = -20^\circ 2'.7$	(6)	$M = -20^\circ 2'.7$	(7)	$M = 234^\circ 52'.0$	
$b = 105^\circ 14'.8$	(10)	$b = 105^\circ 14'.8$	(11)	$c = 43^\circ 17'.2$	
$b - M = 125^\circ 17'.5$	(11)	$b - M = 125^\circ 17'.5$	(12)	$c - M = -191^\circ 34'.8$	
$\log \cos (b - M)$		$\log \sin (b - M)$		$\log \sin (c - M)$	
$= 9.76173 \text{ } n$	(12)	$= 9.91180$	(13)	$= 9.30263$	
$\log m = 9.88923$	(9)	$\log m = 9.88923$	(10)	$\log m = 9.65989$	
$\log \cos a = 9.65096 \text{ } n$	(13)	$\cos \sin a = 0.03530$	(4)	$\cos \sin a = 0.03530$	
$a = 116^\circ 35'.7$	(14)	$\cos \sin c = 0.16389$	(14)	$\cos \sin b = 0.01556$	
		$\log \cot \gamma = 0.00022$	(15)	$\log \cot \beta = 9.01338$	
		$\gamma = 44^\circ 59'.1$	(16)	$\beta = 84^\circ 6'.7$	

$$2. \quad a = 103^\circ 44'.7, \quad b = 64^\circ 12'.3, \quad \gamma = 98^\circ 33'.8.$$

$$\therefore M = 211^\circ 19'.8, \quad c = 103^\circ 30'.6, \quad a = 98^\circ 56'.0; \quad M = -17^\circ 7'.4, \quad \beta = 66^\circ 18'.0.$$

$$3. \quad a = 156^\circ 12'.2, \quad b = 112^\circ 48'.6, \quad \gamma = 76^\circ 32'.4.$$

$$\therefore M = 174^\circ 8'.4, \quad c = 63^\circ 48'.9, \quad a = 154^\circ 4'.2; \quad M = 151^\circ 2'.3, \quad \beta = 87^\circ 27'.1.$$

152. Case IV. Given Two Angles and the Included Side (α, β, c).—From (4) and (5), Art. 147, we have

$$\tan \frac{1}{2} (a - b) = \tan \frac{1}{2} c \frac{\sin \frac{1}{2} (\alpha - \beta)}{\sin \frac{1}{2} (\alpha + \beta)}, \quad (1)$$

$$\tan \frac{1}{2} (a + b) = \tan \frac{1}{2} c \frac{\cos \frac{1}{2} (\alpha - \beta)}{\cos \frac{1}{2} (\alpha + \beta)}. \quad (2)$$

$$\text{Then} \quad a = \frac{1}{2} (a + b) + \frac{1}{2} (a - b),$$

$$b = \frac{1}{2} (a + b) - \frac{1}{2} (a - b).$$

To obtain γ , use (6) and (7), Art. 147 :

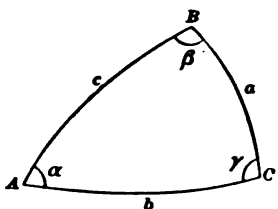


FIG. 119.

$$\cot \frac{1}{2} \gamma = \tan \frac{1}{2} (\alpha - \beta) \frac{\sin \frac{1}{2} (a + b)}{\sin \frac{1}{2} (a - b)}, \quad (3)$$

$$\cot \frac{1}{2} \gamma = \tan \frac{1}{2} (\alpha + \beta) \frac{\cos \frac{1}{2} (a + b)}{\cos \frac{1}{2} (a - b)}. \quad (4)$$

The agreement of the values of $\frac{1}{2} \gamma$ found from (3) and (4) is a check upon the computation. The sine proportion may also be used as a check.

See the note, Art. 150.

1. Solve the triangle when $\alpha = 104^\circ 30'.7$, $\beta = 62^\circ 52'.1$, $c = 56^\circ 6'.4$.

$\alpha = 104^\circ 30'.7$	$\log \tan \frac{1}{2} c = 9.72665$	$\log \tan \frac{1}{2} c = 9.72665$
$\beta = 62^\circ 52'.1$	$\log \sin \frac{1}{2} (\alpha - \beta) = 9.55079$	$\log \cos \frac{1}{2} (\alpha - \beta) = 9.97066$
$\frac{1}{2} (\alpha + \beta) = 83^\circ 41'.4$	$\text{col } \sin \frac{1}{2} (\alpha + \beta) = 0.00234$	$\text{col } \cos \frac{1}{2} (\alpha + \beta) = 0.95897$
$\frac{1}{2} (\alpha - \beta) = 20^\circ 49'.3$	$\log \tan \frac{1}{2} (a - b) = 9.28008$	$\log \tan \frac{1}{2} (a + b) = 0.65628$
$\frac{1}{2} c = 28^\circ 3'.2$	$\frac{1}{2} (a - b) = 10^\circ 47'.4$	$\frac{1}{2} (a + b) = 77^\circ 33'.4$
		$\frac{1}{2} (a - b) = 10^\circ 47'.4$
$\log \tan \frac{1}{2} (\alpha - \beta) = 9.58012$	$\log \tan \frac{1}{2} (a + \beta) = 0.95633$	
$\log \sin \frac{1}{2} (a + b) = 9.98968$	$\log \cos \frac{1}{2} (a + b) = 9.33339$	$a = 88^\circ 20'.8$
$\text{col } \sin \frac{1}{2} (a - b) = 0.72768$	$\text{col } \cos \frac{1}{2} (a - b) = 0.00775$	$b = 66^\circ 46'.0$
$\log \cot \frac{1}{2} \gamma = 0.29748$	$\log \cot \frac{1}{2} \gamma = 0.29747$	
$\frac{1}{2} \gamma = 26^\circ 45'.2$	$\frac{1}{2} \gamma = 26^\circ 45'.2$	

2. $\alpha = 140^\circ 43'.2$, $\beta = 100^\circ 4'.6$, $c = 60^\circ 43'.6$.

$$\therefore \frac{1}{2} (a + b) = 132^\circ 38'.88; \frac{1}{2} (a - b) = 13^\circ 16'.32; a = 145^\circ 55'.20;$$

$$b = 119^\circ 22'.56; \frac{1}{2} \gamma = 40^\circ 7'.42.$$

3. $\alpha = 140^\circ 24'.6$, $\beta = 12^\circ 18'.6$, $c = 28^\circ 7'.7$.

$$\therefore \frac{1}{2} (a + b) = 24^\circ 55'.9; \frac{1}{2} (a - b) = 13^\circ 3'.0; a = 37^\circ 58'.9;$$

$$b = 11^\circ 52'.9; \frac{1}{2} \gamma = 14^\circ 36'.7.$$

153. Case IV. Second Method. Given β , γ , α , to find One Element only.

(1) To find a only.

$$\cos a = -\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a.$$

$$\text{Let } \left. \begin{aligned} m \sin M &= \sin \gamma \cos a, \\ m \cos M &= \cos \gamma. \end{aligned} \right\} \quad (1)$$

$$\therefore \cos a = -m (\cos \beta \cos M - \sin \beta \sin M).$$

$$\therefore \cos a = -m \cos (M + \beta). \quad (2)$$

(2) To find one side only, b or c .—Permuting (6), Art. 124,

$$\sin \beta \cot \gamma = \cot c \sin a - \cos a \cos \beta.$$

$$\therefore \cot c = \frac{\sin \beta \cot \gamma + \cos a \cos \beta}{\sin a} = \frac{\sin \beta \cos \gamma + \cos a \cos \beta \sin \gamma}{\sin a \sin \gamma}.$$

$$\text{Let} \quad \left. \begin{aligned} m \sin M &= \sin \gamma \cos a, \\ m \cos M &= \cos \gamma. \end{aligned} \right\} \quad (3)$$

$$\therefore \cot c = \frac{m \sin (M + \beta)}{\sin a \sin \gamma} \quad (4)$$

The formula for $\cot b$ may be found by permuting b and c in (3) and (4).

$$1. \quad \beta = 140^\circ 43'.2, \quad \gamma = 100^\circ 4'.6, \quad a = 60^\circ 43'.6.$$

To find a .		To find c .		To find b .	
$\log \sin \gamma = 9.99325$	(1)	$\log \sin \gamma =$	(1)	$\log \sin \beta = 9.80148$	
$\log \cos a = 9.68929$	(3)	$\log \cos a =$	(3)	$\log \cos a = 9.68929$	
$\log (m \sin M)$		$\log (m \sin M)$		$\log (m \sin M)$	
$= 9.68254$	(4)	$=$	(5)	$= 9.49077$	
$\log \sin M = 9.97306$	(7)	$\log \sin M =$	(8)	$\log \sin M = 9.56977$	
$\log \cos M = 9.53348 \text{ n}$	(8)	$\log \cos M =$	(9)	$\log \cos M = 9.96778 \text{ n}$	
$\log \cos \gamma = 9.24296 \text{ n}$	(2)	$\log \cos \gamma =$	(2)	$\log \cos \beta = 9.88877 \text{ n}$	
$\log \tan M = 0.43958 \text{ n}$	(5)	$\log \tan M = 0.43958 \text{ n}$	(6)	$\log \tan M = 9.60200 \text{ n}$	
$M = 109^\circ 58'.4$	(6)	$M = 109^\circ 58'.4$	(7)	$M = 158^\circ 12'.1$	
$\beta = 140^\circ 43'.2$	(10)	$\beta = 140^\circ 43'.2$	(11)	$\gamma = 100^\circ 4'.6$	
$M + \beta = 250^\circ 41'.6$	(11)	$M + \beta = 250^\circ 41'.6$	(12)	$M + \gamma = 258^\circ 16'.7$	
$\log \cos (M + \beta)$		$\log \sin (M + \beta)$		$\log \sin (M + \gamma)$	
$= 9.51933 \text{ n}$	(12)	$= 9.97486 \text{ n}$	(13)	$= 9.99085 \text{ n}$	
$\log (-m) = 9.70948 \text{ n}$	(9)	$\log m = 9.70948$	(10)	$\log m = 9.92099$	
$\log \cos a = 9.22881$	(18)	$\cos \sin a = 0.05934$	(4)	$\cos \sin a = 0.05934$	
$a = 80^\circ 15'.0$	(14)	$\cos \sin \gamma = 0.00675$	(14)	$\cos \sin \beta = 0.19852$	
		$\log \cot c = 9.75043 \text{ n}$	(15)	$\log \cot b = 0.16970 \text{ n}$	
		$c = 119^\circ 22'.5$	(16)	$b = 145^\circ 55'.2$	

$$2. \quad a = 104^\circ 30'.7, \quad \beta = 62^\circ 52'.1, \quad c = 56^\circ 6'.4.$$

$$\therefore M = 114^\circ 53'.9, \quad \gamma = 53^\circ 30'.5, \quad a = 88^\circ 20'.8; \quad M = 47^\circ 25'.2, \quad b = 66^\circ 46'.1.$$

$$3. \quad a = 140^\circ 24'.6, \quad \beta = 12^\circ 18'.6, \quad c = 28^\circ 7'.7.$$

$$\therefore M = 143^\circ 53'.7, \quad \gamma = 29^\circ 13'.3, \quad a = 37^\circ 58'.8; \quad M = 10^\circ 53'.6, \quad b = 11^\circ 52'.9.$$

$$4. \quad a = 109^\circ 23'.5, \quad \beta = 76^\circ 47'.4, \quad c = 121^\circ 32'.8.$$

$$\therefore M = 236^\circ 4'.1, \quad \gamma = 113^\circ 51'.9, \quad a = 118^\circ 28'.5; \quad M = 294^\circ 9'.8, \quad b = 65^\circ 7'.5.$$

154. Case V. Given Two Sides and the Angle Opposite One of them (a, b, α). — Find β by the sine proportion,

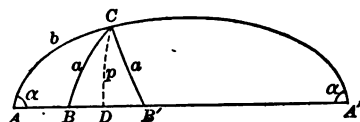


Fig. 120.

$$\sin \beta = \sin b \frac{\sin \alpha}{\sin a} \quad (1)$$

Find c by (4) and (5), Art. 147,

$$\tan \frac{1}{2} c = \tan \frac{1}{2} (a - b) \frac{\sin \frac{1}{2} (\alpha + \beta)}{\sin \frac{1}{2} (\alpha - \beta)}, \quad (2)$$

$$\tan \frac{1}{2} c = \tan \frac{1}{2} (a + b) \frac{\cos \frac{1}{2} (\alpha + \beta)}{\cos \frac{1}{2} (\alpha - \beta)}. \quad (3)$$

Find γ by (6) and (7), Art. 147,

$$\cot \frac{1}{2} \gamma = \tan \frac{1}{2} (\alpha - \beta) \frac{\sin \frac{1}{2} (\alpha + b)}{\sin \frac{1}{2} (\alpha - b)}, \quad (4)$$

$$\cot \frac{1}{2} \gamma = \tan \frac{1}{2} (\alpha + \beta) \frac{\cos \frac{1}{2} (\alpha + b)}{\cos \frac{1}{2} (\alpha - b)}. \quad (5)$$

The agreement of the values of $\frac{1}{2} c$ and of $\frac{1}{2} \gamma$ is a check upon the computation.

Since β is found by means of its sine, it may be either in the first or in the second quadrant; hence there may be two solutions. If b differs more than a from 90° , β must be of the same species as b , and the quadrant in which β lies is fixed. But if b does not differ more than a from 90° , we cannot determine by the first rule for species the quadrant in which β must lie, and both values of β may be admissible. Hence, in-

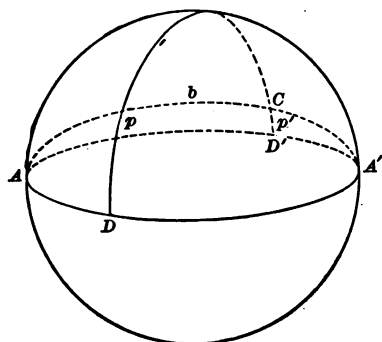


Fig. 121.

spect for two solutions when the side opposite the required angle differs less from 90° than the side opposite the given angle. After finding β , the second rule of Art. 146 will show whether both values are admissible.

1. Solve the triangle when $a = 148^\circ 34'.4$, $b = 142^\circ 11'.6$, $\alpha = 153^\circ 17'.6$. Since b differs less than a from 90° , there may be two solutions.

$\log \sin b = 9.78746$	$\frac{1}{2}(a+b) = 145^\circ 23'.0$	$\frac{1}{2}(a-b) = 3^\circ 11'.4$
$\log \sin a = 9.65265$	$\frac{1}{2}(a+\beta) = 92^\circ 35'.65$	$\frac{1}{2}(a+\beta') = 150^\circ 41'.95$
$\text{col sin } a = 0.28282$	$\frac{1}{2}(a-\beta) = 60^\circ 41'.95$	$\frac{1}{2}(a-\beta') = 2^\circ 35'.65$
$\log \sin \beta = 9.72293$		
$\beta = 31^\circ 53'.7$		
$\beta' = 148^\circ 6'.3$		

The second rule for species is satisfied for both β and β' ; hence there are two solutions.

	First.	or	Second.
$\log \tan \frac{1}{2}(a-b)$	$= 8.74612$		8.74612
$\log \sin \frac{1}{2}(a+\beta)$	$= 9.99955$		9.68966
$\text{col sin } \frac{1}{2}(a-\beta)$	$= 0.05945$		1.34427
$\log \tan \frac{1}{2}c$	$= 8.80512$		9.78006
$\frac{1}{2}c$	$= 3^\circ 39'.18$		$31^\circ 4'.46$
c	$= 7^\circ 18'.36$		$c' = 62^\circ 8'.92$
$\log \tan \frac{1}{2}(a+b)$	$= 9.83903$	n	9.83903
$\log \cos \frac{1}{2}(a+\beta)$	$= 8.65573$	n	9.94055
$\text{col cos } \frac{1}{2}(a-\beta)$	$= 0.31034$		0.00045
$\log \tan \frac{1}{2}c$	$= 8.80510$		9.78003
$\frac{1}{2}c$	$= 3^\circ 39'.17$		$31^\circ 4'.39$
c	$= 7^\circ 18'.34$		$c' = 62^\circ 8'.78$
$\log \tan \frac{1}{2}(a-\beta)$	$= 0.25089$	or	8.65617
$\log \sin \frac{1}{2}(a+b)$	$= 9.75441$		9.75441
$\text{col sin } \frac{1}{2}(a-b)$	$= 1.25456$		1.25456
$\log \cot \frac{1}{2}\gamma$	$= 1.25986$		9.66514
$\frac{1}{2}\gamma$	$= 3^\circ 8'.79$		$65^\circ 10'.68$
γ	$= 6^\circ 17'.58$		$\gamma' = 130^\circ 21'.36$
$\log \tan \frac{1}{2}(a+\beta)$	$= 1.34383$	n	9.74911
$\log \cos \frac{1}{2}(a+b)$	$= 0.91538$	n	9.91538
$\text{col cos } \frac{1}{2}(a-b)$	$= 0.00067$		0.00067
$\log \cot \frac{1}{2}\gamma$	$= 1.25988$		9.66516
$\frac{1}{2}\gamma$	$= 3^\circ 8'.78$		$65^\circ 10'.62$
γ	$= 6^\circ 17'.56$		$\gamma' = 130^\circ 21'.24$

2. $a = 40^\circ 20'.4$, $b = 20^\circ 18'.2$, $\alpha = 60^\circ 44'.4$.

$\therefore \beta = 27^\circ 52'.9$; $\frac{1}{2}c = 23^\circ 34'.34$; $\frac{1}{2}\gamma = 49^\circ 26'.7$.

3. $a = 98^\circ 16'$, $b = 74^\circ 38'$, $\alpha = 78^\circ 40'$.

$\therefore \beta = 72^\circ 49'.25$; $\frac{1}{2}c = 75^\circ 53'.0$ or $75^\circ 52'.6^*$; $\frac{1}{2}\gamma = 76^\circ 1'.5$ or $76^\circ 1'.1^*$.

155. Case V. Second Method. Given a, b, α , to find One Element only.

(1) To find β only.

$$\sin \beta = \frac{\sin b}{\sin a} \sin \alpha. \quad (1)$$

* These values would be taken, since a small error in β will affect them less than if they had been computed from the other formulas.

(2) *To find c only.*

$$\cos b \cos c + \sin b \sin c \cos a = \cos a.$$

$$\text{Let } \left. \begin{aligned} m \sin M &= \sin b \cos a, \\ m \cos M &= \cos b \end{aligned} \right\} \quad (2)$$

$$\therefore m \cos (M - c) = \cos a.$$

$$\therefore \cos (M - c) = \frac{\cos a}{m}. \quad (3)$$

$M - c$ may be either in the first and fourth quadrants or in the second and third; if there are two solutions both values of $M - c$ will give $c < 180^\circ$.

(3) *To find γ only.* — From (7), Art. 124,

$$\cos b \cos \gamma + \sin \gamma \cot a = \cot a \sin b.$$

$$\therefore \cos b \sin a \cos \gamma + \sin \gamma \cos a = \cot a \sin b \sin a.$$

$$\text{Let } \left. \begin{aligned} m \sin M &= \cos b \sin a, \\ m \cos M &= \cos a. \end{aligned} \right\} \quad (4)$$

$$\therefore m \sin (M + \gamma) = \cot a \sin b \sin a.$$

$$\therefore \sin (M + \gamma) = \frac{\cot a \sin b \sin a}{m}. \quad (5)$$

$M + \gamma$ may be either in the first and second quadrants or in the third and fourth; if there are two solutions, both values of $M + \gamma$ will give $\gamma < 180^\circ$.

$$1. \ a = 148^\circ 34'.4, \ b = 142^\circ 11'.6, \ a = 153^\circ 17'.6.$$

To find c .

$$\begin{aligned} \log \sin b &= 9.78746 & (1) \\ \log \cos a &= 9.95101 \ n & (3) \\ \log (m \sin M) &= 9.73847 \ n & (4) \\ \log \sin M &= 9.75561 \ n & (7) \\ \log \cos M &= 9.91481 \ n & (8) \\ \log \cos b &= 9.89767 \ n & (2) \\ \log \tan M &= 9.84080 & (5) \\ M &= 214^\circ 43'.6 & (6) \\ \text{colog } m &= 0.01714 & (9) \\ \log \cos a &= 9.93111 \ n & (10) \\ \log \cos (M - c) &= 9.94825 \ n & (11) \\ M - c &= 152^\circ 34'.8 & (12) \\ M &= 214^\circ 43'.6 & (14) \\ M - c' &= 207^\circ 25'.2 & (13) \\ \therefore c &= 62^\circ 8'.8 & (15) \\ \text{and } c' &= 7^\circ 18'.4 & (16) \end{aligned}$$

Two values.

To find γ .

$$\begin{aligned} \log \cos b &= 9.89767 \ n & (1) \\ \log \sin a &= 9.65265 & (3) \\ \log (m \sin M) &= 9.55032 \ n & (5) \\ \log \sin M &= 9.56746 \ n & (8) \\ \log \cos M &= 9.96815 \ n & (9) \\ \log \cos a &= 9.95101 \ n & (4) \\ \log \tan M &= 9.59931 & (6) \\ M &= 201^\circ 40'.6 & (7) \\ \text{colog } m &= 0.01714 & (10) \\ \log \cot a &= 0.21393 \ n & (11) \\ \log \sin b &= 9.78746 & (2) \\ \log \sin a &= 9.65265 & (3) \\ \log \sin (M + \gamma) &= 9.67118 \ n & (12) \\ M + \gamma &= 207^\circ 58'.2 & (13) \\ M &= 201^\circ 40'.6 & (15) \\ M + \gamma' &= 332^\circ 1'.8 & (14) \\ \therefore \gamma &= 6^\circ 17'.6 & (16) \\ \text{and } \gamma' &= 130^\circ 21'.2 & (17) \end{aligned}$$

Two values.

$$2. a = 40^\circ 20'.4, b = 20^\circ 18'.2, \alpha = 60^\circ 44'.4.$$

$$\therefore M = 10^\circ 15'.0, c = 47^\circ 8'.7; M = 59^\circ 8'.8, \gamma = 98^\circ 53'.5.$$

$$3. a = 98^\circ 16', b = 74^\circ 38', \alpha = 78^\circ 40'.$$

$$\therefore M = 35^\circ 34'.0, c = 151^\circ 45'.4; M = 52^\circ 53'.9, \gamma = 152^\circ 2'.5.$$

156. Case VI. Given Two Angles and the Side Opposite One of them (α, β, a). — Find b by the sine proportion,

$$\sin b = \sin \beta \frac{\sin a}{\sin \alpha} \quad (1)$$

Find c by (4) and (5), Art. 147,

$$\tan \frac{1}{2} c = \tan \frac{1}{2} (a - b) \frac{\sin \frac{1}{2} (\alpha + \beta)}{\sin \frac{1}{2} (\alpha - \beta)}, \quad (2)$$

$$\tan \frac{1}{2} c = \tan \frac{1}{2} (a + b) \frac{\cos \frac{1}{2} (\alpha + \beta)}{\cos \frac{1}{2} (\alpha - \beta)}. \quad (3)$$

Find γ by (6) and (7), Art. 147,

$$\cot \frac{1}{2} \gamma = \tan \frac{1}{2} (\alpha - \beta) \frac{\sin \frac{1}{2} (a + b)}{\sin \frac{1}{2} (a - b)}, \quad (4)$$

$$\cot \frac{1}{2} \gamma = \tan \frac{1}{2} (\alpha + \beta) \frac{\cos \frac{1}{2} (a + b)}{\cos \frac{1}{2} (a - b)}. \quad (5)$$

The agreement of the values of $\frac{1}{2} c$ and of $\frac{1}{2} \gamma$ is a check upon the computation.

Since b is found by means of its sine, it may be either in the first or in the second quadrant; hence there may be two solutions. If β differs more than α from 90° , β and b must be of the same species, and the quadrant in which b lies is fixed. But if β does not differ more than α from 90° , we cannot determine by the first rule for species the quadrant in which b must lie, and both values of b may be admissible. Hence, *inspect for two solutions when the angle opposite the required side differs less from 90° than the angle opposite the given side.* After finding b , the second rule of Art. 146 will show whether both values are admissible.

1. Solve the triangle when $\alpha = 143^\circ 17'.4$, $\beta = 70^\circ 18'.4$, $a = 160^\circ 40'.6$. Since β differs less than α from 90° , there may be two solutions.

$\log \sin \alpha = 9.51969$	$\frac{1}{2} (\alpha + \beta) = 106^\circ 47'.9$	$\frac{1}{2} (\alpha - \beta) = 36^\circ 29'.5$
$\log \sin \beta = 9.97383$	$\frac{1}{2} (\alpha + b) = 96^\circ 2'.65$	$\frac{1}{2} (\alpha + b') = 154^\circ 37'.95$
$\text{col } \sin \alpha = 0.22347$	$\frac{1}{2} (\alpha - b) = 64^\circ 37'.95$	$\frac{1}{2} (\alpha - b') = 6^\circ 2'.65$
$\log \sin b = 9.71699$		

$$b = 31^\circ 24'.7$$

$$b' = \underline{148^\circ 35'.3}$$

The second rule for species is satisfied for both b and b' ; hence there are two solutions.

	First.		Second.
$\log \tan \frac{1}{2} (a - b) =$	0.32409	or	9.02483
$\log \sin \frac{1}{2} (a + \beta) =$	9.98106		9.98106
$\text{col sin } \frac{1}{2} (a - \beta) =$	0.22570		0.22570
$\log \tan \frac{1}{2} c =$	0.53085		9.23159
$\frac{1}{2} c =$	73° 35'.28		9° 40'.38
$c =$	147° 10'.56	$c' =$	19° 20'.76
$\log \tan \frac{1}{2} (a + b) =$	0.97517 n	or	9.67591 n
$\log \cos \frac{1}{2} (a + \beta) =$	9.46091 n		9.46091 n
$\text{col cos } \frac{1}{2} (a - \beta) =$	0.09478		0.09478
$\log \tan \frac{1}{2} c =$	0.53086		9.23160
$\frac{1}{2} c =$	73° 35'.30		9° 40'.39
$c =$	147° 10'.60	$c' =$	19° 20'.78
$\log \tan \frac{1}{2} (a - \beta) =$	9.86908	or	9.86908
$\log \sin \frac{1}{2} (a + b) =$	9.99758		9.63187
$\text{col sin } \frac{1}{2} (a - b) =$	0.04403		0.97759
$\log \cot \frac{1}{2} \gamma =$	9.91069		0.47854
$\frac{1}{2} \gamma =$	50° 51'.00		18° 22'.74
$\gamma =$	101° 42'.00	$\gamma' =$	36° 45'.48
$\log \tan \frac{1}{2} (a + \beta) =$	0.52016 n	or	0.52016 n
$\log \cos \frac{1}{2} (a + b) =$	9.02241 n		9.95597 n
$\text{col cos } \frac{1}{2} (a - b) =$	0.36813		0.00242
$\log \cot \frac{1}{2} \gamma =$	9.91070		0.47855
$\frac{1}{2} \gamma =$	50° 50'.96		18° 22'.71
$\gamma =$	101° 41'.92	$\gamma' =$	36° 45'.42

2. $\alpha = 117^\circ 54'.4$, $\beta = 45^\circ 8'.6$, $\alpha = 76^\circ 37'.5$.

$\therefore b = 51^\circ 17'.9$; $\frac{1}{2} c = 20^\circ 32'.3$ or $20^\circ 32'.4$; $\frac{1}{2} \gamma = 18^\circ 19'.4$.

3. $\alpha = 104^\circ 40'.0$, $\beta = 80^\circ 13'.6$, $\alpha = 126^\circ 50'.4$.

$\therefore b = 54^\circ 36'.8$; $\frac{1}{2} c = 73^\circ 48'.4$ or $73^\circ 48'.5$; $\frac{1}{2} \gamma = 69^\circ 49'.5$ or $69^\circ 49'.6$;

and $b' = 125^\circ 23'.2$; $\frac{1}{2} c' = 3^\circ 25'.6$ or $3^\circ 25'.5$; $\frac{1}{2} \gamma' = 4^\circ 8'.8$.

157. Case VI. Second Method. Given α , β , α , to find One Element only.

(1) To find b only.

$$\sin b = \frac{\sin \beta}{\sin \alpha} \sin \alpha. \quad (1)$$

(2) To find c only. — Permuting (3), Art. 124,

$$\cot a \sin c - \cos c \cos \beta = \sin \beta \cot a.$$

$$\therefore \cos a \sin c - \sin a \cos c \cos \beta = \sin a \sin \beta \cot a.$$

$$\text{Let } \left. \begin{aligned} m \sin M &= \sin a \cos \beta, \\ m \cos M &= \cos a. \end{aligned} \right\} \quad (2)$$

$$\therefore m \sin (c - M) = \sin a \sin \beta \cot a.$$

$$\therefore \sin (c - M) = \frac{\sin a \sin \beta \cot a}{m} \quad (3)$$

$c - M$ may be either in the first and second quadrants, or in the third and fourth; if there are two solutions, both values of $c - M$ will give $c < 180^\circ$.

(3) *To find γ only.*

$$-\cos \beta \cos \gamma + \sin \beta \sin \gamma \cos a = \cos a.$$

$$\text{Let } \left. \begin{aligned} m \sin M &= \cos a \sin \beta, \\ m \cos M &= \cos \beta. \end{aligned} \right\} \quad (4)$$

$$\therefore m \cos (M + \gamma) = -\cos a.$$

$$\therefore \cos (M + \gamma) = -\frac{\cos a}{m} \quad (5)$$

$M + \gamma$ may be either in the first and fourth quadrants, or in the second and third; if there are two solutions, both values of $M + \gamma$ will give $\gamma < 180^\circ$.

$$1. \alpha = 143^\circ 17'.4, \beta = 70^\circ 18'.4, a = 160^\circ 40'.6.$$

To find c .

$$\log \sin a = 9.51969 \quad (1)$$

$$\log \cos \beta = 9.52761 \quad (3)$$

$$\log (m \sin M) = 9.04730 \quad (5)$$

$$\log \sin M = 9.06947 \quad (8)$$

$$\log \cos M = 9.99699 \quad (9)$$

$$\log \cos a = 9.97482 \quad (2)$$

$$\log \tan M = 9.07248 \quad (6)$$

$$M = 173^\circ 15'.7 \quad (7)$$

$$\text{colog } m = 0.02217 \quad (10)$$

$$\log \sin a = 9.51969 \quad (1)$$

$$\log \sin \beta = 9.97383 \quad (4)$$

$$\log \cot a = 0.12746 \quad (11)$$

$$\log \sin (c - M) = 9.64315 \quad (12)$$

$$c - M = 206^\circ 5'.1 \quad (13)$$

$$M = 173^\circ 15'.7 \quad (15)$$

$$c' - M = 338^\circ 54'.9 \quad (14)$$

$$c = 19^\circ 20'.8 \quad (16)$$

$$c' = 147^\circ 10'.6 \quad (17)$$

Two values.

To find γ .

$$\log \cos a = 9.97482 \quad n \quad (1)$$

$$\log \sin \beta = 9.97383 \quad (2)$$

$$\log (m \sin M) = 9.94865 \quad n \quad (4)$$

$$\log \sin M = 9.97082 \quad n \quad (7)$$

$$\log \cos M = 9.54977 \quad (8)$$

$$\log \cos \beta = 9.52761 \quad (3)$$

$$\log \tan M = 0.42104 \quad n \quad (5)$$

$$M = 290^\circ 46'.2 \quad (6)$$

$$\text{colog } (-m) = 0.02217 \quad n \quad (9)$$

$$\log \cos a = 9.90400 \quad n \quad (10)$$

$$\log \cos (M + \gamma) = 9.92617 \quad (11)$$

$$M + \gamma = 32^\circ 28'.2 \quad (12)$$

$$M = 290^\circ 46'.2 \quad (14)$$

$$M + \gamma' = 327^\circ 31'.8 \quad (13)$$

$$\gamma = 101^\circ 42'.0 \quad (15)$$

$$\gamma' = 36^\circ 45'.6 \quad (16)$$

Two values.

2. $\alpha = 117^\circ 54'.4$, $\beta = 45^\circ 8'.6$, $\gamma = 76^\circ 37'.5$.
 $\therefore M = 71^\circ 22'.3$, $c = 41^\circ 4'.9$; $M = 13^\circ 5'.3$, $\gamma = 36^\circ 38'.8$.
3. $\alpha = 104^\circ 40'.0$, $\beta = 80^\circ 13'.6$, $\gamma = 126^\circ 50'.4$.
 $\therefore M = 167^\circ 14'.0$, $c = 147^\circ 38'.9$ or $6^\circ 51'.1$;
 $M = -73^\circ 58'.3$, $\gamma = 139^\circ 39'.0$ or $8^\circ 17'.6$.

OBLIQUE TRIANGLES SOLVED BY RIGHT TRIANGLES.

158. **General Method.** — From any vertex C of the triangle draw an arc p of a great circle perpendicular to the opposite

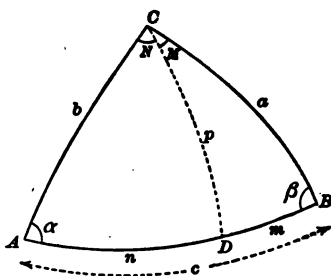


FIG. 122.

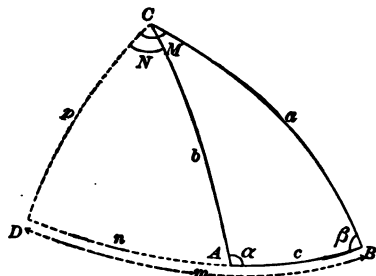


FIG. 123.

side, dividing the triangle into two right triangles. Denote the segments of the side by m and n , and the corresponding segments of the angle by M and N .

The opposite side must in some cases be produced to meet the perpendicular arc, as in Fig. 123. The segments of the side are AD and DB , and their signs are so taken that their algebraic sum shall be equal to the side; that is, *if a segment is entirely exterior to the triangle, it is negative*.

The perpendicular p may have either of two supplemental values; we shall always place it in the same quadrant as its opposite angle in the triangle first used in the solution, in accordance with the rule for species.

159. **Special Formula.** — To prove

$$\tan \frac{1}{2}(m+n) \tan \frac{1}{2}(m-n) = \tan \frac{1}{2}(\alpha+\beta) \tan \frac{1}{2}(\alpha-\beta). \quad (1)$$

In both Fig. 122 and Fig. 123, by Napier's rules,

$$\cos a = \cos m \cos p, \quad \text{and} \quad \cos b = \cos n \cos p.$$

$$\therefore \cos p = \frac{\cos a}{\cos m} = \frac{\cos b}{\cos n}$$

$$\therefore \frac{\cos m}{\cos n} = \frac{\cos a}{\cos b}$$

$$\therefore \frac{\cos m - \cos n}{\cos m + \cos n} = \frac{\cos a - \cos b}{\cos a + \cos b},$$

which becomes, from (4) and (3), Art. 73,

$$\tan \frac{1}{2}(m+n) \tan \frac{1}{2}(m-n) = \tan \frac{1}{2}(a+b) \tan \frac{1}{2}(a-b). \quad \text{Q.E.D.}$$

160. Case I. Given a, b, c .—From (1), Art. 159, we have

$$\tan \frac{1}{2}(m-n) = \tan \frac{1}{2}(a+b) \tan \frac{1}{2}(a-b) \cot \frac{1}{2}c, \quad (1)$$

since $m+n=c$. We shall consider $\frac{1}{2}(m-n)$ as being numerically less than 90° , so that it will be a negative angle when its tangent is negative. After $\frac{1}{2}(m-n)$ has been found, we have

$$\left. \begin{aligned} m &= \frac{1}{2}c + \frac{1}{2}(m-n), \\ n &= \frac{1}{2}c - \frac{1}{2}(m-n). \end{aligned} \right\} \quad (2)$$

A negative value of m or of n indicates that the segment, and hence the corresponding triangle, is exterior to the given triangle. Note that m is always measured from the side that is called a , and n from b .

In the triangles ACD and DCB we now know the two sides, so that the other elements can be found by Napier's rules. The example shows the method of finding the elements of the original triangle from the results of the computation.

$$1. \ a = 114^\circ 43'.3, \ b = 136^\circ 19'.6, \ c = 43^\circ 18'.5.$$

From (1), $\frac{1}{2}(m-n) = 33^\circ 56'.81$, whence $m = 55^\circ 36'.06$, $n = -12^\circ 17'.56$. The negative value of n shows that ACD is exterior to the triangle.

$$\text{From } BCD \text{ we find } DBC = \beta = 132^\circ 15'.3, \ DCB = M = 65^\circ 17'.0.$$

From ACD we find $DAC = 180^\circ - \alpha = 103^\circ 11'.6$, $ACD = N = -17^\circ 57'.5$, giving N the negative sign since it is exterior to the triangle. Hence

$$\alpha = 76^\circ 48'.4; \ \gamma = M + N = 47^\circ 19'.5.$$

$$2. \ a = 76^\circ 40'.4, \ b = 54^\circ 21'.3, \ c = 36^\circ 8'.7.$$

$$\therefore \frac{1}{2}(m-n) = 53^\circ 0'.38; \ m = 71^\circ 4'.73; \ n = -34^\circ 56'.03; \ \beta = 46^\circ 17'.3;$$

$$M = 76^\circ 27'.0; \ N = -44^\circ 48'.2; \ \gamma = 31^\circ 38'.8; \ \alpha = 120^\circ 3'.6.$$

$$3. \alpha = 124^\circ 34'.9, \beta = 66^\circ 7'.2, c = 109^\circ 43'.5.$$

$$\therefore \frac{1}{2}(m-n) = -76^\circ 37'.32; m = -21^\circ 45'.57; n = +131^\circ 29'.07; \beta = 74^\circ 1'.7;$$

$$M = -26^\circ 45'.6; \alpha = 120^\circ 2'.7; N = +124^\circ 59'.2; \gamma = 98^\circ 13'.6.$$

$$4. \alpha = 30^\circ 17'.6, \beta = 22^\circ 14'.4, c = 18^\circ 51'.8.$$

$$\therefore m = 21^\circ 14'.6; n = -2^\circ 22'.8; \beta = 48^\circ 17'.1; M = 45^\circ 54'.8;$$

$$\alpha = 95^\circ 50'.0; N = -6^\circ 17'.9; \gamma = 39^\circ 36'.9.$$

$$5. \alpha = 130^\circ 46'.0, \beta = 113^\circ 21'.4, c = 102^\circ 16'.2.$$

$$\therefore \frac{1}{2}(m-n) = -11^\circ 8'.6; m = 39^\circ 59'.5; n = 62^\circ 16'.7; \beta = 136^\circ 19'.25;$$

$$M = 58^\circ 3'.4; \alpha = 145^\circ 15'.9; N = 74^\circ 37'.75; \gamma = 132^\circ 41'.2.$$

161. Case II. Given α, β, γ . — Apply the method of Case I to the polar triangle, and thence find the elements of the original triangle.

$$1. \alpha = 116^\circ 19'.4, \beta = 83^\circ 19'.2, \gamma = 106^\circ 10'.6.$$

In the polar triangle,

$$\alpha' = 63^\circ 40'.6, \beta' = 96^\circ 40'.8, c' = 73^\circ 49'.4.$$

$$\therefore \frac{1}{2}(m' - n') = -66^\circ 18'.1, m' = -29^\circ 23'.4, n' = +108^\circ 12'.8.$$

The negative value of m' shows that $B'C'D'$ is exterior to the triangle.

From $B'C'D'$ we find

$$D'B'C' = 180^\circ - \beta' = 73^\circ 49'.2, D'C'B' = M' = -33^\circ 11'.8,$$

giving M' the negative sign since it is exterior to the triangle.

From $A'C'D'$ we find

$$D'A'C' = \alpha' = 60^\circ 4'.7, N' = +101^\circ 25'.5.$$

$$\therefore \beta' = 106^\circ 10'.8, \gamma' = M' + N' = 68^\circ 13'.7.$$

Passing from the polar to the original triangle,

$$\alpha = 119^\circ 55'.3; \beta = 73^\circ 49'.2; c = 111^\circ 46'.3.$$

$$2. \alpha = 110^\circ 36'.4, \beta = 122^\circ 8'.7, \gamma = 140^\circ 20'.3.$$

$$\therefore \frac{1}{2}(m' - n') = 29^\circ 27'.90; m' = 49^\circ 17'.75; n' = -9^\circ 38'.05;$$

$$\beta' = 64^\circ 4'.9; M' = 54^\circ 5'.4; \alpha' = 96^\circ 7'.4; N' = -11^\circ 24'.0; \gamma' = 42^\circ 41'.4;$$

$$\therefore \alpha = 83^\circ 52'.6, \beta = 115^\circ 55'.1, c = 137^\circ 18'.6.$$

$$3. \alpha = 120^\circ 50'.6, \beta = 78^\circ 6'.1, \gamma = 81^\circ 12'.3.$$

$$\therefore \frac{1}{2}(m' - n') = -63^\circ 33'.19; m' = -14^\circ 9'.34; n' = 112^\circ 57'.04;$$

$$\beta' = 98^\circ 39'.7; M' = -16^\circ 33'.0; \alpha' = 60^\circ 9'.6;$$

$$N' = 109^\circ 46'.0; \gamma' = 93^\circ 13'.0;$$

$$\therefore \alpha = 119^\circ 50'.4, \beta = 81^\circ 20'.3, c = 86^\circ 47'.0.$$

$$4. \alpha = 80^\circ 20'.2, \beta = 73^\circ 46'.7, \gamma = 54^\circ 8'.5.$$

$$\therefore \frac{1}{2}(m' - n') = 7^\circ 15'.69; m' = 70^\circ 11'.44; n' = 55^\circ 40'.06;$$

$$\beta' = 118^\circ 12'.7; M' = 72^\circ 37'.5; \alpha' = 115^\circ 12'.8;$$

$$N' = 59^\circ 19'.1; \gamma' = 131^\circ 56'.6;$$

$$\therefore \alpha = 64^\circ 47'.2, \beta = 61^\circ 47'.3, c = 48^\circ 3'.4.$$

$$5. \alpha = 100^\circ 51'.3, \beta = 80^\circ 47'.6, \gamma = 74^\circ 3'.3.$$

$$\therefore \frac{1}{2}(m' - n') = -83^\circ 50'.76; m' = -30^\circ 52'.41; n' = 130^\circ 49'.11;$$

$$\beta' = 96^\circ 35'.0; M' = -31^\circ 30'.0; \alpha' = 81^\circ 15'.1;$$

$$N' = 136^\circ 6'.8; \gamma' = 104^\circ 36'.8;$$

$$\therefore \alpha = 98^\circ 44'.9, b = 83^\circ 25'.0, c = 75^\circ 23'.2.$$

162. Case III. Given α, b, γ . — From the end of one of the sides, as b , let fall an arc of a great circle perpendicular to the other side. In the triangle DAC we know b and γ ; hence we find n, N , and p by Napier's rules, considering p as of the same species as γ . Then $m = \alpha - n$, being negative when $n > \alpha$, showing that the triangle BAD is then exterior to the triangle BAC .

Now in the triangle BAD we know DB and AD , and we find c, M , and ABD by Napier's rules.

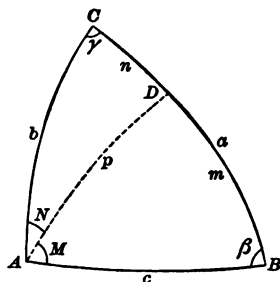


Fig. 124.

$$1. \alpha = 105^\circ 14'.8, b = 43^\circ 17'.2, \gamma = 112^\circ 47'.4.$$

$$\therefore n = 159^\circ 57'.3, N = 150^\circ 0'.4, p = 140^\circ 47'.53.$$

$$\therefore m = -54^\circ 42'.5, \text{ showing that } BAD \text{ is exterior to } BAC.$$

In the triangle BAD we find

$$ABD = 180^\circ - \beta = 135^\circ 0'.8, c = 116^\circ 35'.6, M = -65^\circ 53'.7,$$

giving M the negative sign since it is exterior to the triangle.

$$\text{Hence } \beta = 44^\circ 59'.2, \alpha = N + M = 84^\circ 6'.7.$$

$$2. \alpha = 103^\circ 44'.7, b = 64^\circ 12'.3, \gamma = 98^\circ 33'.8.$$

$$\therefore N = 160^\circ 54'.7; n = 162^\circ 52'.6; p = 117^\circ 5'.1; m = -59^\circ 7'.9;$$

$$c = 103^\circ 30'.6; \beta = 66^\circ 18'.0; M = -61^\circ 58'.7; \alpha = 98^\circ 56'.0.$$

$$3. \alpha = 156^\circ 12'.2, b = 112^\circ 48'.6, \gamma = 76^\circ 32'.4.$$

$$\therefore N = 148^\circ 18'.6; n = 151^\circ 2'.3; p = 63^\circ 41'.8; m = 5^\circ 9'.9;$$

$$c = 63^\circ 43'.8; \beta = 87^\circ 27'.1; M = 5^\circ 45'.5; \alpha = 154^\circ 4'.1.$$

163. Case IV. Given α, β, c . — Let fall from the vertex of one of the angles, as $\alpha = BAC$ (Fig. 124), an arc of a great circle perpendicular to the opposite side. In the triangle ABD we know c and β , and we find m, M , and p by Napier's rules, considering p as of the same species as β . Then $N = \alpha - M$, a negative value of N showing that the point D lies on BC produced, the triangle ACD being then exterior to the given triangle.

In the triangle ACD we now know p and CAD , and we find b , γ , and π by Napier's rules.

$$1. \alpha = 140^\circ 43'.2, \beta = 100^\circ 4'.6, c = 60^\circ 43'.6.$$

$$\therefore m = 162^\circ 39'.9, p = 120^\circ 48'.86, M = 160^\circ 1'.7.$$

$$\text{Then } N = \alpha - M = -19^\circ 18'.5.$$

$$\therefore b = 119^\circ 22'.5, ACD = 180^\circ - \gamma = 99^\circ 45'.1, \pi = -16^\circ 44'.8,$$

giving π the negative sign since it is exterior to the triangle.

$$\therefore \gamma = 80^\circ 14'.9, a = m + \pi = 145^\circ 55'.1.$$

$$2. \alpha = 104^\circ 30'.7, \beta = 62^\circ 52'.1, c = 56^\circ 6'.4.$$

$$\therefore M = 42^\circ 34'.8; N = 61^\circ 55'.9; m = 34^\circ 10'.2; p = 47^\circ 37'.5;$$

$$b = 66^\circ 46'.0; \gamma = 53^\circ 30'.4; \pi = 54^\circ 10'.7; a = 88^\circ 20'.9.$$

$$3. \alpha = 140^\circ 24'.6, \beta = 12^\circ 18'.3, c = 28^\circ 7'.7.$$

$$\therefore M = 79^\circ 6'.4; N = 61^\circ 18'.2; m = 27^\circ 34'.7; p = 5^\circ 46'.1;$$

$$b = 11^\circ 52'.9; \gamma = 29^\circ 13'.3; \pi = 10^\circ 24'.3; a = 37^\circ 59'.0.$$

164. Case V. Given a, b, α . — Let fall an arc of a great circle from the intersection of a and b , perpendicular to c . In this case there will be two solutions if a is intermediate in value between p and both b and $180^\circ - b$ (Art. 120).

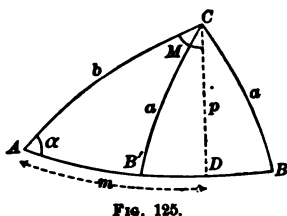


FIG. 125.

In the triangle ACD , knowing b and α , find m , M , and p by Napier's rules. Then in the triangle DCB , knowing p and a , find DB , DCB , and DBC . Then in the triangle ACB we have

$$c = AB = m + DB, \gamma = ACB = M + DCB, \beta = DBC;$$

and in the triangle ACB' ,

$$c' = AB' = m - DB, \gamma' = ACB' = M - DCB, \beta' = 180^\circ - DBC.$$

$$1. \alpha = 148^\circ 34'.4, b = 142^\circ 11'.6, a = 153^\circ 17'.6.$$

$$\therefore p = 164^\circ 0'.52, \text{ and there are two solutions.}$$

$$m = 34^\circ 43'.5, M = 68^\circ 19'.4.$$

$$\text{Also, } DB = 27^\circ 25'.1, DBC = 148^\circ 6'.3, DCB = 62^\circ 1'.8.$$

$$\therefore c = 62^\circ 8'.6, \gamma = 130^\circ 21'.2, \beta = 148^\circ 6'.3,$$

$$\text{and } c' = 7^\circ 18'.4, \gamma' = 6^\circ 17'.6, \beta' = 31^\circ 53'.7.$$

2. $\alpha = 40^\circ 20'.4$, $b = 20^\circ 18'.2$, $\alpha = 60^\circ 44'.4$.

$\therefore p = 17^\circ 37'.3$; $m = 10^\circ 15'.0$; $M = 30^\circ 51'.2$; $\beta = 27^\circ 52'.9$;

$DB = 36^\circ 53'.7$; $DCB = 68^\circ 2'.3$; $c = 47^\circ 8'.7$; $\gamma = 98^\circ 53'.5$.

3. $\alpha = 98^\circ 18'$, $b = 74^\circ 38'$, $\alpha = 78^\circ 40'$.

$\therefore p = 70^\circ 59'.25$; $m = 35^\circ 34'.0$; $M = 37^\circ 6'.1$; $\beta = 72^\circ 49'.25$;

$DB = 116^\circ 11'.4$; $DCB = 114^\circ 58'.4$; $c = 151^\circ 45'.4$; $\gamma = 152^\circ 2'.5$.

165. Case VI. Given α , β , α .—Pass to the polar triangle, in which we shall know α' , b' , and α' , and solve by the method of Art. 164. There may be two solutions of the polar triangle, and therefore of the triangle itself.

1. $\alpha = 143^\circ 17'.4$, $\beta = 70^\circ 18'.4$, $\alpha = 160^\circ 40'.6$.

$\therefore \alpha' = 36^\circ 42'.6$, $b' = 109^\circ 41'.6$, $\alpha' = 19^\circ 19'.4$.

$\therefore p' = 18^\circ 9'.13$, and there will be two solutions.

$M' = 96^\circ 44'.3$, $m' = 110^\circ 46'.3$.

Also $D'B' = 32^\circ 28'.25$, $D'C'B' = 68^\circ 54'.9$, $D'B'C' = 31^\circ 24'.7$.

$\therefore c_1' = 143^\circ 14'.55$, $\gamma_1' = 160^\circ 39'.2$, $\beta_1' = 31^\circ 24'.7$,

and $c_1'' = 78^\circ 18'.05$, $\gamma_1'' = 32^\circ 49'.4$, $\beta_1'' = 148^\circ 35'.3$.

Taking the supplements to obtain the elements of the original triangle,

$\gamma = 36^\circ 45'.45$, $c = 19^\circ 20'.8$, $b = 148^\circ 35'.3$,

and $\gamma' = 101^\circ 41'.95$, $c' = 147^\circ 10'.6$, $b' = 31^\circ 24'.7$.

2. $\alpha = 117^\circ 54'.4$, $\beta = 45^\circ 8'.6$, $\alpha = 76^\circ 37'.5$.

$\therefore p' = 136^\circ 23'.8$; $M' = 18^\circ 37'.7$; $m' = 13^\circ 5'.3$;

$D'C'B' = 120^\circ 17'.5$; $D'B'C' = 128^\circ 42'.1$; $D'B' = 130^\circ 15'.9$;

$\gamma' = 138^\circ 55'.2$; $c' = 143^\circ 21'.2$;

$\therefore b = 51^\circ 17'.9$; $c = 41^\circ 4'.8$; $\gamma = 36^\circ 38'.8$.

3. $\alpha = 104^\circ 40'.0$, $\beta = 80^\circ 13'.6$, $\alpha = 126^\circ 50'.4$.

$\therefore p' = 52^\circ 3'.8$; $M' = 102^\circ 46'.0$; $m' = 106^\circ 1'.7$;

$D'C'B' = 70^\circ 22'.9$; $D'B'C' = 54^\circ 36'.8$; $D'B' = 65^\circ 40'.7$;

$\gamma_1' = 172^\circ 8'.9$; $\gamma_1'' = 32^\circ 23'.1$; $c_1' = 171^\circ 42'.4$;

$c_1'' = 40^\circ 21'.0$; $\beta_1' = 54^\circ 36'.8$; $\beta_1'' = 125^\circ 23'.2$.

$\therefore c = 7^\circ 51'.1$; $\gamma = 8^\circ 17'.6$; $b = 125^\circ 23'.2$;

and $c' = 147^\circ 36'.9$; $\gamma' = 139^\circ 39'.0$; $b' = 54^\circ 36'.8$.

CHAPTER XII.

APPLICATIONS OF SPHERICAL TRIGONOMETRY.

166. To find the Shortest Distance between Two Points on the Surface of the Earth,* the earth being treated as a sphere. — North latitudes and west longitudes are considered positive. Let QQ' be the equator, P the north pole, A and B the two

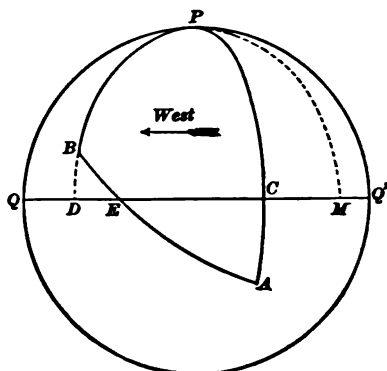


FIG. 126.

points, and PM the meridian from which the longitudes are measured. The longitude of A is MPC and that of B is MPD , both being positive since they are measured westward. The latitudes are CA and DB , the former being negative since it is measured southward.

In the triangle APB the sides AP and BP are found by algebraically subtracting the latitudes from 90° , and the angle APB is the algebraic difference of the longitudes. Hence we know two sides and their included angle, so that we can solve the triangle, using the method of Art. 151 when the distance only is required, and that of Art. 150 when we wish to find all the elements.

1. Find the shortest distance between New York, $40^\circ 45'.4$ N., $73^\circ 58'.4$ W., and Rio Janeiro, $22^\circ 54'.4$ S., $43^\circ 10'.4$ W.

$\therefore BP = 49^\circ 14'.6$, $AP = 112^\circ 54'.4$, $APB = 30^\circ 48'.0$. *Ans.* $AB = 69^\circ 48'.2$.

2. Find the shortest distance between New York, $40^\circ 45'.4$ N., $73^\circ 58'.4$ W., and Paris, $48^\circ 50'.2$ N., $2^\circ 20'.2$ E. *Ans.* $AB = 52^\circ 26'.8$.

* The shortest distance between two points on a sphere is the arc of the great circle passing through the points.

If the bearings of the great circle AB at A and B are required, it will be necessary to find the angles PAB and PBA .

3. A ship sailed from Calcutta, $22^{\circ} 34'.8$ N., $88^{\circ} 27'.3$ E., on an arc of a great circle to Melbourne, $37^{\circ} 48'.0$ S., $144^{\circ} 58'.0$ E. Find the distance sailed and the bearings* at both points.

Ans. At Calcutta, S. $41^{\circ} 56'.61$ E.; at Melbourne, S. $51^{\circ} 21'.47$ E.; distance, $80^{\circ} 22'.4$ or $80^{\circ} 22'.6$.

4. A ship sailed from the Cape of Good Hope, $34^{\circ} 22'$ S., $18^{\circ} 29'$ E., on an arc of a great circle to Cape St. Roque, $5^{\circ} 28'$ S., $35^{\circ} 16'$ W. Find the distance sailed and the bearings* at both points.

Ans. At G. H., N. $72^{\circ} 28'.0$ W.; at S. R., N. $52^{\circ} 15'.0$ W.; distance, $57^{\circ} 20'.4$.

5. A ship sailed from Bombay, $18^{\circ} 56'$ N., $72^{\circ} 53'$ E., on an arc of a great circle to the Cape of Good Hope, $34^{\circ} 22'$ S., $18^{\circ} 29'$ E. Find the distance sailed and the bearings* at both points.

Ans. At Bombay, S. $44^{\circ} 12'.8$ W.; at G. H., S. $53^{\circ} 2'.6$ W.; distance, $74^{\circ} 15'.2$ or $74^{\circ} 15'.4$.

6. A ship sailed from Bombay, $18^{\circ} 56'$ N., $72^{\circ} 53'$ E., on an arc of a great circle for the Cape of Good Hope, $34^{\circ} 22'$ S., $18^{\circ} 29'$ E. Find the distance to the equator and the bearing* and longitude at the equator. [Use the triangle BDE ; the angle $PBA = 135^{\circ} 47'.2$ was found in Ex. 5.]

Ans. S. $41^{\circ} 18'.1$ W.; distance, $25^{\circ} 34'.5$; longitude, $55^{\circ} 21'.8$ E.

7. From a point whose latitude is 17° N. and longitude 130° W. a ship sailed an arc of a great circle over a distance of 4150 miles, starting S. $54^{\circ} 20'$ W. Find its latitude and longitude if the length of 1° is $69\frac{1}{4}$ miles.

Ans. Lat., $19^{\circ} 40'.52$ or $19^{\circ} 40'.60$ S.; Long., $178^{\circ} 20'.9$ W.

167. Given the Lengths of the Three Edges of a Parallelopiped that meet in a Point, and the Angles between them, to find the Surface and the Volume of the Parallelopiped. — Let OG be the solid, AD the perpendicular from A to BOC , and hence AOD a plane perpendicular to BOC . Let the angles and edges be

$$BOC = a, AOC = b,$$

$$AOB = c, OA = l,$$

$$OB = m, OC = n.$$

Describe a sphere with a radius of unity about O as a center, its intersections with the planes forming the figure marked by the primed letters.

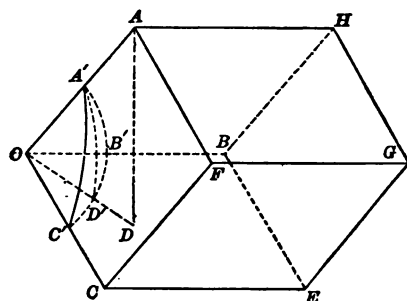


FIG. 127.

* The course of the ship.

Then the surface is

$$\begin{aligned} S &= 2 OBEC + 2 OAF C + 2 OBHA \\ &= 2 (mn \sin a + ln \sin b + lm \sin c). \end{aligned} \quad (1)$$

In the triangle $A'D'B'$, right-angled at D' , we have

$$\sin D'A' = \sin B'A' \sin A'B'D';$$

$$\therefore \sin D'A' = \sin c \sin A'B'D'.$$

But in the triangle $A'B'C'$ we know the three sides a, b, c ; hence

$$\begin{aligned} \sin A'B'D' &= 2 \sin \frac{1}{2} A'B'D' \cos \frac{1}{2} A'B'D' \\ &= \frac{2}{\sin a \sin c} \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}. \end{aligned}$$

$$\therefore \sin D'A' = \sin DOA$$

$$= \frac{2}{\sin a} \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}.$$

$$\therefore DA = OA \sin DOA$$

$$= \frac{2l}{\sin a} \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}.$$

Hence the volume is

$$\begin{aligned} V &= OBEC \times DA \\ &= 2lmn \sqrt{\sin s \sin (s-a) \sin (s-b) \sin (s-c)}. \end{aligned} \quad (2)$$

168. To find the Volume of a Regular Polyhedron.—Let AB be the edge in which two adjacent faces intersect, D its middle point, C and E the centers of the polygonal faces, and O the center of the sphere inscribed in the polyhedron, the faces being tangent to the sphere at C and E . Then

$$DC = DE; DA = DB;$$

$$CDA = CDB = EDA = EDB = 90^\circ;$$

$$DCO = 90^\circ; DEO = 90^\circ.$$

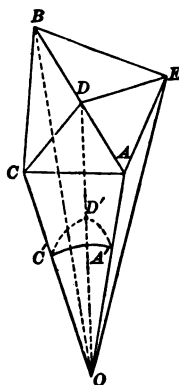


FIG. 128.

Let a = length of an edge AB ,

s = number of sides of each polygonal face,

n = number of faces meeting at a vertex of the polyhedron,

N = number of faces of the polyhedron,

E = edge angle CDE of the polyhedron.

$$\text{Then } CD = AD \cot ACD = \frac{1}{2} a \cot \frac{180^\circ}{s}.$$

$$CO = CD \tan CDO = CD \tan \frac{1}{2} E.$$

$$\therefore CO = \frac{1}{2} a \cot \frac{180^\circ}{s} \tan \frac{1}{2} E. \quad (1)$$

About O as a center, with a unit radius, describe a sphere, and let its intersections with the three planes form the triangle $A'C'D'$. Then

$$A'C'D' = ACD = \frac{180^\circ}{s}; \quad A'D'C' = 90^\circ; \quad C'A'D' = \frac{1}{2} \frac{360^\circ}{n}.$$

By Napier's rules,

$$\cos C'A'D' = \cos C'D' \sin A'C'D',$$

$$\text{or } \cos \frac{180^\circ}{n} = \cos C'D' \sin \frac{180^\circ}{s}.$$

But

$$\cos C'D' = \cos COD = \cos (90^\circ - CDO) = \sin CDO = \sin \frac{1}{2} E.$$

$$\therefore \cos \frac{180^\circ}{n} = \sin \frac{1}{2} E \sin \frac{180^\circ}{s}.$$

$$\therefore \sin \frac{1}{2} E = \cos \frac{180^\circ}{n} \operatorname{cosec} \frac{180^\circ}{s}. \quad (2)$$

Then, if A is the area of a face, the volume is

$$V = \frac{1}{3} CO \times A \times N = \frac{1}{24} N s a^3 \cot^2 \frac{180^\circ}{s} \tan \frac{1}{2} E. \quad (3)$$

Find $\frac{1}{2} E$ from (2) and then V from (3).

1. Dodecahedron, formed by 12 regular pentagons, 3 meeting at a vertex.

$$\begin{aligned} \therefore s = 5, n = 3, N = 12. \quad & \log \cos 60^\circ = 9.69897 & \log \frac{Ns}{24} = 0.39794 \\ & \log \operatorname{cosec} 36^\circ = 0.23078 & \log \cot^2 36^\circ = 0.27748 \\ & \log \sin \frac{1}{2} E = 9.92975 & \log \tan \frac{1}{2} E = 0.20896 \\ & & 0.88438 \\ \therefore V = 7.663 a^3. \end{aligned}$$

2. Tetrahedron, formed by 4 equilateral triangles, 3 meeting at a vertex.

$$\therefore s = 3, n = 3, N = 4. \quad \text{Ans. } V = 0.1179 a^3.$$

3. Cube, formed by 6 squares, 3 meeting at a vertex.

$$\therefore s = 4, n = 3, N = 6. \quad \text{Ans. } V = a^3.$$

4. Octahedron, formed by 8 equilateral triangles, 4 meeting at a vertex.

$$\therefore s = 3, n = 4, N = 8. \quad \text{Ans. } V = 0.4714 a^3.$$

5. Icosahedron, formed by 20 equilateral triangles, 5 meeting at a vertex.

$$\therefore s = 3, n = 5, N = 20. \quad \text{Ans. } V = 2.182 a^3.$$

169. If from Any Point in a Trirectangular Triangle Arcs of Great Circles are drawn to the Vertices,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1,$$

where α, β , and γ are the arcs. — In Fig. 129, produce YP and ZP to D and E . In the right triangle PDX ,

$$\sin PD = \sin \alpha \sin PXD; \therefore \cos \beta = \sin \alpha \sin PXD. \quad (1)$$

In the right triangle PXE ,

$$\sin PE = \sin \alpha \sin PXE; \therefore \cos \gamma = \sin \alpha \cos PXD. \quad (2)$$

Squaring (1) and (2), and adding, we have

$$\cos^2 \beta + \cos^2 \gamma = \sin^2 \alpha.$$

$$\therefore \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

Q.E.D.

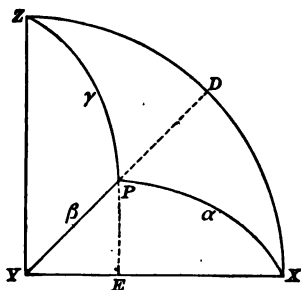


FIG. 129.

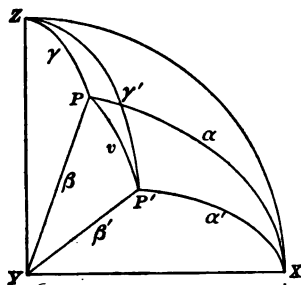


FIG. 130.

170. If from Any Two Points P and P' in a Trirectangular Triangle Arcs of Great Circles are drawn to the Three Vertices, and if v is the Length of the Arc PP' , prove that

$$\cos v = \cos \alpha \cos \alpha' + \cos \beta \cos \beta' + \cos \gamma \cos \gamma'.$$

In the triangle PYP' (Fig. 130),

$$\cos v = \cos \beta \cos \beta' + \sin \beta \sin \beta' \cos PYP'. \quad (1)$$

But

$$\cos PYP' = \cos (ZYP' - ZYP).$$

$$\therefore \cos PYP' = \cos ZYP' \cos ZYP + \sin ZYP' \sin ZYP. \quad (2)$$

$$\left. \begin{aligned} \text{In } ZYP, \cos \gamma &= \sin \beta \cos ZYP.* \\ \text{In } ZYP', \cos \gamma' &= \sin \beta' \cos ZYP'.* \\ \text{In } XYP, \cos \alpha &= \sin \beta \cos XYP* = \sin \beta \sin ZYP. \\ \text{In } XYP', \cos \alpha' &= \sin \beta' \cos XYP'.* = \sin \beta' \sin ZYP'. \end{aligned} \right\} \quad (3)$$

Substituting in (1) the values found from (2) and (3),

$$\cos v = \cos \beta \cos \beta' + \cos \gamma \cos \gamma' + \cos \alpha \cos \alpha'. \quad \text{Q.E.D.}$$

This is the formula for the cosine of the angle between two lines in space, the angles made by them with three lines at right angles to each other being α, β, γ , and α', β', γ' , respectively.

171. To find the Angle α' between the Chords of Two Sides of a Spherical Triangle, having given the Two Sides b and c , and the Angle α between them. — Let $AB = c, AC = b$, the spherical angle $BAC = \alpha$, and the plane angle $BAC = \alpha', O$ being the center of the sphere. About A as a center describe a sphere, and let its intersections with the planes OAB, OAC , and BAC form the triangle DEF . Then

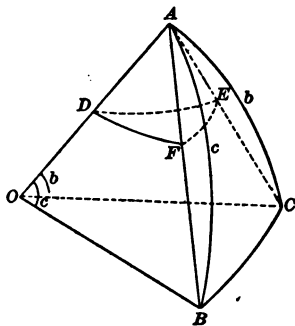


FIG. 181.

$$DF = OAB = 90^\circ - \frac{1}{2}c;$$

$$DE = OAC = 90^\circ - \frac{1}{2}b;$$

$$FDE = \alpha; FE = BAC = \alpha'.$$

$$\therefore \cos FE = \cos DE \cos DF + \sin DE \sin DF \cos FDE.$$

$$\therefore \cos \alpha' = \sin \frac{1}{2}b \sin \frac{1}{2}c + \cos \frac{1}{2}b \cos \frac{1}{2}c \cos \alpha. \quad (1)$$

This formula is true for all values of b, c , and α . When b and c are small, the *correction* that must be applied to α to obtain α' may be found from (1) as follows:

Let $p = b + c$, and $q = b - c$. Then, from Art. 72,

$$\begin{aligned} \cos \alpha' &= \frac{1}{2} \cos \frac{1}{2}q - \frac{1}{2} \cos \frac{1}{2}p + \frac{1}{2} (\cos \frac{1}{2}p + \cos \frac{1}{2}q) \cos \alpha \\ &= -\sin^2 \frac{1}{4}q + \sin^2 \frac{1}{4}p + (1 - \sin^2 \frac{1}{4}p - \sin^2 \frac{1}{4}q) \cos \alpha \\ &= (\sin^2 \frac{1}{4}p - \sin^2 \frac{1}{4}q) (\sin^2 \frac{1}{2}\alpha + \cos^2 \frac{1}{2}\alpha) + \cos \alpha \\ &\quad - (\sin^2 \frac{1}{4}p + \sin^2 \frac{1}{4}q) (\cos^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\alpha). \end{aligned}$$

$$\therefore \cos \alpha' = \cos \alpha - 2 \sin^2 \frac{1}{4}q \cos^2 \frac{1}{2}\alpha + 2 \sin^2 \frac{1}{4}p \sin^2 \frac{1}{2}\alpha. \quad (2)$$

* Eq. (2), Art. 121.

Let $\alpha' = \alpha + \theta$, where θ is so small that we may place

$$\sin \theta = \theta, \text{ and } \cos \theta = 1.$$

$$\therefore \cos \alpha' = \cos \alpha \cos \theta - \sin \alpha \sin \theta.$$

$$\therefore \cos \alpha' = \cos \alpha - \theta \sin \alpha. \quad (3)$$

Comparing (2) and (3),

$$2\theta \sin \frac{1}{2}\alpha \cos \frac{1}{2}\alpha = 2\sin^2 \frac{1}{4}q \cos^2 \frac{1}{2}\alpha - 2\sin^2 \frac{1}{4}p \sin^2 \frac{1}{2}\alpha.$$

$$\therefore \theta = \sin^2 \frac{1}{4}q \cot \frac{1}{2}\alpha - \sin^2 \frac{1}{4}p \tan \frac{1}{2}\alpha.$$

$$\therefore \theta'' = \frac{1}{\sin 1''} \sin^2 \frac{1}{4}q \cot \frac{1}{2}\alpha - \frac{1}{\sin 1''} \sin^2 \frac{1}{4}p \tan \frac{1}{2}\alpha, \quad (4)$$

since $\theta = \theta'' \sin 1''$ (Art. 81).

172. The Angles of Elevation of Two Points, in the Directions OA and OB , above a Horizontal Plane, and the Inclined Angle AOB , were measured with a Sextant. Find the Horizontal

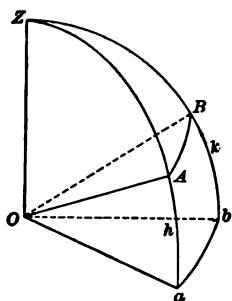


FIG. 182.

Angle between the Points, as seen from O . — Let OZ be the vertical line, Oab the horizontal plane; $aOA = h$, and $bOB = k$ the measured altitudes; and $AOB = c$ the inclined angle. Describe a sphere about O as a center. Then in the triangle AZB ,

$AZ = 90^\circ - h$, $BZ = 90^\circ - k$, $AB = c$, and hence the required angle $aOb = AZB$ may be computed, since we know the three sides of the triangle.

When h and k are small, the correction to be applied to the measured value c to obtain aOb may be found as follows:*

From (2), Art. 121,

$$\cos AZB = \frac{\cos c - \sin h \sin k}{\cos h \cos k} = \frac{\cos c - hk}{(1 - \frac{1}{2}h^2)(1 - \frac{1}{2}k^2)} \quad (\text{Art. 78})$$

$$= \frac{\cos c - hk}{1 - \frac{1}{2}(h^2 + k^2)} = (\cos c - hk) \left[1 + \frac{1}{2}(h^2 + k^2) \right].$$

$$\therefore \cos AZB = \cos c + \frac{1}{2}(h^2 + k^2) \cos c - hk. \quad (1)$$

* Neglecting powers of h and k above the second.

Let θ be the correction to c so that $AZB = c + \theta$.

$$\therefore \cos AZB = \cos c \cos \theta - \sin c \sin \theta.$$

$$\therefore \cos AZB = \cos c - \theta \sin c. \quad (2)$$

Comparing (1) and (2),

$$\theta = - \frac{(h^2 + k^2)(\cos^2 \frac{1}{2}c - \sin^2 \frac{1}{2}c) - 2hk(\cos^2 \frac{1}{2}c + \sin^2 \frac{1}{2}c)}{4 \sin \frac{1}{2}c \cos \frac{1}{2}c}.$$

$$\therefore \theta = \frac{1}{4}(h + k)^2 \tan \frac{1}{2}c - \frac{1}{4}(h - k)^2 \cot \frac{1}{2}c, \quad (3)$$

where θ , h , and k are expressed in circular measure. To find θ in seconds, let $\theta = \theta'' \sin 1''$, $h = h'' \sin 1''$, $k = k'' \sin 1''$.

$$\therefore \theta'' = \frac{1}{4}(h'' + k'')^2 \sin 1'' \tan \frac{1}{2}c - \frac{1}{4}(h'' - k'')^2 \sin 1'' \cot \frac{1}{2}c. \quad (4)$$

SPHERICAL EXCESS.

173. Area of a Spherical Triangle.—From geometry we know that the areas of any two triangles are to each other as their spherical excesses, the spherical excess being the amount by which the sum of the three angles exceeds 180° . We also know that the area of the trirectangular triangle is $\frac{1}{2}\pi r^2$, and that its spherical excess is 90° . If A is the area of any triangle, and E its spherical excess expressed in degrees, we have

$$A : \frac{1}{2}\pi r^2 = E : 90^\circ. \quad (1)$$

$$\therefore A = E \frac{\pi r^2}{180^\circ}, \quad (2)$$

and
$$E = A \frac{180^\circ}{\pi r^2}. \quad (3)$$

174. Lhuillier's Theorem.—We have

$$\begin{aligned} \tan \frac{1}{4}E &= \frac{\sin \frac{1}{4}(\alpha + \beta + \gamma - \pi)}{\cos \frac{1}{4}(\alpha + \beta + \gamma - \pi)} \cdot \frac{2 \cos \frac{1}{4}(\alpha + \beta + \pi - \gamma)}{2 \cos \frac{1}{4}(\alpha + \beta + \pi - \gamma)} \\ &= \frac{\sin \frac{1}{2}(\alpha + \beta) - \sin \frac{1}{2}(\pi - \gamma)}{\cos \frac{1}{2}(\alpha + \beta) + \cos \frac{1}{2}(\pi - \gamma)}, \end{aligned}$$

from (6) and (7), Art. 72.

$$\therefore \tan \frac{1}{4}E = \frac{\sin \frac{1}{2}(\alpha + \beta) - \cos \frac{1}{2}\gamma}{\cos \frac{1}{2}(\alpha + \beta) + \sin \frac{1}{2}\gamma}.$$

Hence, from (1) and (2), Art. 145, substituting for $\sin \frac{1}{2}(\alpha + \beta)$ and $\cos \frac{1}{2}(\alpha + \beta)$, we have

$$\begin{aligned}\tan \frac{1}{4} E &= \frac{\cos \frac{1}{2}(a - b) - \cos \frac{1}{2}c}{\cos \frac{1}{2}(a + b) + \cos \frac{1}{2}c} \cdot \frac{\cos \frac{1}{2}\gamma}{\sin \frac{1}{2}\gamma} \\ &= \frac{\sin \frac{1}{4}(a - b + c) \sin \frac{1}{4}(b + c - a)}{\cos \frac{1}{4}(a + b + c) \cos \frac{1}{4}(a + b - c)} \cot \frac{1}{2}\gamma,\end{aligned}$$

from (4) and (3), Art. 73.

$$\begin{aligned}\therefore \tan \frac{1}{4} E &= \frac{\sin \frac{1}{2}(s - b) \sin \frac{1}{2}(s - a)}{\cos \frac{1}{2}s \cos \frac{1}{2}(s - c)} \sqrt{\frac{\sin s \sin(s - c)}{\sin(s - a) \sin(s - b)}} \\ &= \sqrt{\left[\frac{\sin^2 \frac{1}{2}(s - b) \sin^2 \frac{1}{2}(s - a)}{\cos^2 \frac{1}{2}s \cos^2 \frac{1}{2}(s - c)} \times \right.} \\ &\quad \left. \frac{\sin \frac{1}{2}s \cos \frac{1}{2}s \sin \frac{1}{2}(s - c) \cos \frac{1}{2}(s - c)}{\sin \frac{1}{2}(s - a) \cos \frac{1}{2}(s - a) \sin \frac{1}{2}(s - b) \cos \frac{1}{2}(s - b)} \right]} \\ \therefore \tan \frac{1}{4} E &= \sqrt{\tan \frac{1}{2}s \tan \frac{1}{2}(s - a) \tan \frac{1}{2}(s - b) \tan \frac{1}{2}(s - c)}. \text{ Q.E.I.}\end{aligned}$$

175. Spherical Excess in Terms of Two Sides and their Included Angle.

$$\begin{aligned}\tan \frac{1}{2} E &= \frac{\sin \frac{1}{2}(\alpha + \beta + \gamma - \pi)}{\cos \frac{1}{2}(\alpha + \beta + \gamma - \pi)} = \frac{-\cos \frac{1}{2}(\alpha + \beta + \gamma)}{\sin \frac{1}{2}(\alpha + \beta + \gamma)} \\ &= \frac{\sin \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}\gamma - \cos \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}\gamma}{\sin \frac{1}{2}(\alpha + \beta) \cos \frac{1}{2}\gamma + \cos \frac{1}{2}(\alpha + \beta) \sin \frac{1}{2}\gamma}\end{aligned}$$

Substituting for $\sin \frac{1}{2}(\alpha + \beta)$ and $\cos \frac{1}{2}(\alpha + \beta)$ from (1) and (2), Art. 145,

$$\begin{aligned}\tan \frac{1}{2} E &= \frac{\sin \frac{1}{2}\gamma \cos \frac{1}{2}\gamma [\cos \frac{1}{2}(a - b) - \cos \frac{1}{2}(a + b)]}{\cos \frac{1}{2}(a - b) \cos^2 \frac{1}{2}\gamma + \cos \frac{1}{2}(a + b) \sin^2 \frac{1}{2}\gamma} \\ &= \frac{\sin \frac{1}{2}\gamma \cos \frac{1}{2}\gamma [+ 2 \sin \frac{1}{2}a \sin \frac{1}{2}b]}{\frac{1}{2} [\cos \frac{1}{2}(a - b) + \cos \frac{1}{2}(a + b)] + \frac{1}{2} [\cos \frac{1}{2}(a - b) - \cos \frac{1}{2}(a + b)] \cos \gamma} \\ &= \frac{\sin \frac{1}{2}a \sin \frac{1}{2}b \sin \gamma}{\cos \frac{1}{2}a \cos \frac{1}{2}b + \sin \frac{1}{2}a \sin \frac{1}{2}b \cos \gamma} \\ \therefore \tan \frac{1}{2} E &= \frac{\tan \frac{1}{2}a \tan \frac{1}{2}b \sin \gamma}{1 + \tan \frac{1}{2}a \tan \frac{1}{2}b \cos \gamma} \quad \text{Q.E.I.}\end{aligned}$$

176. Approximate Value of the Spherical Excess, neglecting Powers above the Second. — Let the sides of the triangle be so

small that the powers of their circular measures higher than the second may be neglected. We have, from Art. 78,

$$\tan x = x + \frac{1}{3}x^3 + \dots, \quad (1)$$

where x is expressed in circular measure.

Let the lengths of the sides be a, b , and c when expressed in circular measure, and a', b' , and c' in linear measure, r being the radius of the sphere. Then

$$a = \frac{a'}{r}, \quad b = \frac{b'}{r}, \quad c = \frac{c'}{r}. \quad (2)$$

Placing these values of a, b , and c for x in (1), and substituting in Lhuillier's theorem, we have, neglecting powers above the second,

$$\tan \frac{1}{4}E = \sqrt{\frac{s'}{2r} \cdot \frac{s' - a'}{2r} \cdot \frac{s' - b'}{2r} \cdot \frac{s' - c'}{2r}}, \quad (3)$$

where $s' = \frac{1}{2}(a' + b' + c').$ (4)

$$\therefore \tan \frac{1}{4}E = \frac{1}{4r^2} \sqrt{s'(s' - a')(s' - b')(s' - c')}. \quad (5)$$

Since $\frac{1}{4}E$ is small, we place its tangent equal to its arc.

$$\therefore \frac{1}{4}E = \frac{1}{4r^2} \sqrt{s'(s' - a')(s' - b')(s' - c')} \quad (6)$$

or $E = \frac{1}{r^2}A, \quad (7)$

where A is the area of the plane triangle whose sides are a', b' , and c' , E being expressed in circular measure.

To find the value of E in seconds of arc, divide both sides by $\sin 1''$.

$$\therefore \frac{E}{\sin 1''} = E'' = \frac{A}{r^2 \sin 1''}. \quad (8)$$

Hence, whenever the third powers of the circular measures of the sides can be neglected, the spherical excess is found by computing the area of the triangle, considering it as a plane triangle, and dividing the area by $r^2 \sin 1''$.

177. **Approximate Value of the Spherical Excess, neglecting Powers above the Fourth.** — From Lhuillier's theorem,

$$\begin{aligned}\tan^2 \frac{1}{4} E &= \left[\frac{s'}{2r} + \frac{s'^2}{24r^3} \right] \left[\frac{s' - a'}{2r} + \frac{(s' - a')^2}{24r^3} \right] \\ &\quad \left[\frac{s' - b'}{2r} + \frac{(s' - b')^2}{24r^3} \right] \left[\frac{s' - c'}{2r} + \frac{(s' - c')^2}{24r^3} \right] \\ &= \frac{A^2}{16r^4} + \frac{A^2}{192r^6} [(s' - c')^2 + (s' - b')^2 + (s' - a')^2 + s'^2],\end{aligned}$$

where

$$A^2 = s'(s' - a')(s' - b')(s' - c').$$

$$\therefore \tan^2 \frac{1}{4} E = \frac{A^2}{16r^4} + \frac{A^2}{192r^6} (a'^2 + b'^2 + c'^2).$$

$$\begin{aligned}\therefore \tan \frac{1}{4} E &= \frac{A}{4r^2} \left(1 + \frac{a'^2 + b'^2 + c'^2}{12r^2} \right)^{\frac{1}{2}} \\ &= \frac{A}{4r^2} \left(1 + \frac{a'^2 + b'^2 + c'^2}{24r^2} \right).\end{aligned}$$

$$\therefore \frac{1}{4} E'' \sin 1'' = \frac{A}{4r^2} \left(1 + \frac{a'^2 + b'^2 + c'^2}{24r^2} \right).$$

$$\therefore E'' = \frac{A}{r^2 \sin 1''} \left(1 + \frac{a'^2 + b'^2 + c'^2}{24r^2} \right). \quad \text{Q.E.D.}$$

This value exceeds that found in Art. 176 by

$$\frac{A}{r^2 \sin 1''} \cdot \frac{a'^2 + b'^2 + c'^2}{24r^2}.$$

If $a' = b' = c' = 100$ miles, and $r = 3963.3$ miles, we obtain

$$\frac{A}{r^2 \sin 1''} = 56''.863; \quad \frac{a'^2 + b'^2 + c'^2}{24r^2} = 0.00008;$$

so that the correction to the value of E'' given by (8), Art. 176, is only

$$56''.863 \times 0.00008 = 0''.005.$$

178. Legendre's Theorem. — *If the sides of a spherical triangle are very small compared with the radius of the sphere, the angles of the plane triangle whose sides are of the same length as*

those of the spherical triangle, are equal to the corresponding angles of the spherical triangle diminished by one third of the spherical excess. — Let a' , b' , and c' be the lengths of the sides of the spherical triangle expressed in linear measure, and a , b , and c the lengths in circular measure.

$$\therefore a = \frac{a'}{r}, \quad b = \frac{b'}{r}, \quad c = \frac{c'}{r}. \quad (1)$$

Let α be an angle of the spherical triangle and α' the corresponding angle of the plane triangle. We have

$$\cos \alpha = \frac{\cos a - \cos b \cos c}{\sin b \sin c}. \quad (2)$$

From Art. 78,

$$\begin{aligned} \cos a &= 1 - \frac{1}{2}a^2 + \frac{1}{24}a^4 - \dots & \sin b &= b - \frac{1}{6}b^3 + \dots \\ \cos b &= 1 - \frac{1}{2}b^2 + \frac{1}{24}b^4 - \dots & \sin c &= c - \frac{1}{6}c^3 + \dots \\ \cos c &= 1 - \frac{1}{2}c^2 + \frac{1}{24}c^4 - \dots \end{aligned}$$

$$\therefore \cos \alpha = \frac{\frac{1}{2}(b^2 + c^2 - a^2) + \frac{1}{24}(a^4 - b^4 - c^4 - 6b^2c^2)}{bc[1 - \frac{1}{6}(b^2 + c^2)]}, \quad (3)$$

the terms of orders higher than the fourth being neglected.

$$\begin{aligned} \therefore \cos \alpha &= \frac{1}{2bc} [(b^2 + c^2 - a^2) \\ &\quad + \frac{1}{12}(a^4 - b^4 - c^4 - 6b^2c^2)] [1 - \frac{1}{6}(b^2 + c^2)]^{-1} \\ &= \frac{1}{2bc} [(b^2 + c^2 - a^2) \\ &\quad + \frac{1}{12}(a^4 - b^4 - c^4 - 6b^2c^2)] [1 + \frac{1}{6}(b^2 + c^2) + \dots]. \\ \therefore \cos \alpha &= \frac{b^2 + c^2 - a^2}{2bc} + \frac{a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2}{24bc}, \quad (4) \end{aligned}$$

the terms of orders higher than the fourth being neglected, as before.

In the plane triangle,

$$\cos \alpha' = \frac{b'^2 + c'^2 - a'^2}{2b'c'} = \frac{b^2 + c^2 - a^2}{2bc}, \quad (5)$$

from (1)

$$\therefore \cos \alpha = \cos \alpha' + \frac{a'^4 + b'^4 + c'^4 - 2a'^2b'^2 - 2a'^2c'^2 - 2b'^2c'^2}{24b'c'}. \quad (6)$$

$$\therefore \cos \alpha = \cos \alpha' + \frac{1}{r^2} \cdot \frac{a'^4 + b'^4 + c'^4 - 2a'^2b'^2 - 2a'^2c'^2 - 2b'^2c'^2}{24b'c'} \quad (7)$$

Let $s' = \frac{1}{2}(a' + b' + c')$; then

$$\begin{aligned} & s'(s' - a')(s' - b')(s' - c') \\ &= -\frac{1}{16}(a'^4 + b'^4 + c'^4 - 2a'^2b'^2 - 2a'^2c'^2 - 2b'^2c'^2). \end{aligned} \quad (8)$$

But the area of the plane triangle is

$$\sqrt{s'(s' - a')(s' - b')(s' - c')} = \frac{1}{2}b'c' \sin \alpha'. \quad (9)$$

Hence (7) becomes, from (8) and (9),

$$\cos \alpha = \cos \alpha' - \frac{1}{6r^2}b'c' \sin^2 \alpha'. \quad (10)$$

Let $\alpha = \alpha' + \theta. \quad (11)$

$$\therefore \cos \alpha = \cos \alpha' \cos \theta - \sin \alpha' \sin \theta. \quad (12)$$

Since θ is small, we may place $\cos \theta = 1$, and $\sin \theta = \theta$.

$$\therefore \cos \alpha = \cos \alpha' - \theta \sin \alpha'. \quad (13)$$

Comparing (10) and (13),

$$\theta = \frac{1}{6r^2}b'c' \sin \alpha' = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{b'c' \sin \alpha'}{r^2}. \quad (14)$$

Hence, from (7), Art. 176,

$$\theta = \frac{1}{3}E,$$

and, from (11),

$$\alpha' = \alpha - \frac{1}{3}E. \quad \text{Q.E.D.}$$

179. Application of Legendre's Theorem. — In the New York State Survey the angles of the spherical triangle, whose vertices were at Howlett, Gilbertsville, and Eagle, were measured, the distance from Howlett to Gilbertsville having been already computed. The measured values were

At Howlett,	$\alpha = 85^\circ 18' 57''.71$	$\log b = 4.54227 \ 32$
At Eagle,	$\beta = 51^\circ 35' 41''.61$	$\log r = 6.80459 \ 32$
At Gilbertsville,	$\gamma = 43^\circ 5' 24''.24$	

$$\therefore \alpha + \beta + \gamma = 180^\circ \ 0' \ 3''.56$$

The formula for the spherical excess is (Art. 176)

$$E'' = \frac{A}{r^2 \sin 1''} = \frac{1}{2} \cdot \frac{b^2 \sin \alpha \sin \gamma}{\sin \beta} \cdot \frac{1}{r^2 \sin 1''}.$$

$$\begin{aligned}
 \log b^2 &= 9.08455 \\
 \text{colog } 2 &= 9.69897 - 10 \\
 \log \sin \alpha &= 9.99855 - 10 \\
 \log \sin \gamma &= 9.83451 - 10 \\
 \text{col sin } \beta &= 0.10588 \\
 \text{colog } r^2 &= 6.39081 - 20 \\
 \text{col sin } 1'' &= 5.31443 \\
 \log E'' &= 0.42770
 \end{aligned}$$

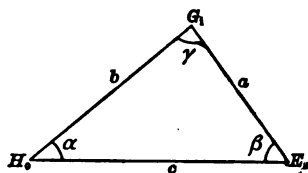


FIG. 133.

$$\begin{aligned}
 \therefore E'' &= 2''.6773; \\
 \therefore \frac{1}{2} E'' &= 0''.8924.
 \end{aligned}$$

The errors due to observation therefore amounted to $3''.56 - 2''.677 = 0''.883$. This discrepancy was distributed among the three angles according to the method of least squares,* giving the following results :

	OBSERVED ANGLES.	CORRECTION.	SPHERICAL ANGLES.	$\frac{1}{2} E''$.	PLANE ANGLES.
$\alpha = 85^\circ 18'$	$57''.71$	$- 0''.747$	$56''.963$	$0''.893$	$56''.070 = \alpha'$
$\beta = 51^\circ 35'$	$41''.61$	$+ 1''.355$	$42''.965$	$0''.892$	$42''.073 = \beta'$
$\gamma = 43^\circ 5'$	$24''.24$	$- 1''.491$	$22''.749$	$0''.892$	$21''.857 = \gamma'$
Sum = $180^\circ 0'$	$3''.56$	$- 0''.883$	$2''.677$	$2''.677$	$0''.000$

Using the plane triangle, we find by the sine proportion :

$$\begin{aligned}
 \log b' &= 4.542 \ 2732 & \log b' &= 4.542 \ 2732 \\
 \text{col sin } \beta' &= 0.105 \ 8837 & \text{col sin } \beta' &= 0.105 \ 8837 \\
 \log \sin \alpha' &= 9.998 \ 5468 - 10 & \log \sin \gamma' &= 9.834 \ 5089 - 10 \\
 \log a' &= 4.646 \ 7037 & \log c' &= 4.482 \ 6658 \\
 a' &= 44330.61 \text{ meters.} & c' &= 30385.46 \text{ meters.}
 \end{aligned}$$

These are the distances between the points measured on the great circles joining them.

ASTRONOMICAL APPLICATIONS.

180. Definitions. — Let us consider the earth as a point O (Fig. 134), and let a sphere be described about O as a center, with a radius indefinitely great, so that all the stars shall be within the sphere. The figure represents the sphere as seen from the outside.

* Eleven angles were involved in the adjustment.

describe small circles about P as the pole, apparently moving in the direction $EQWQ'$.

The equator $EQWQ'$ is the great circle cut from the sphere by a plane through O perpendicular to the axis of the earth.

The *latitude* of the observer is the angular distance QZ from the equator to the zenith. Since $PQ = 90^\circ$ and $ZN = 90^\circ$, we have $NP = QZ$, i.e. the elevation of the pole above the horizon is equal to the latitude of the place.

The *hour circle* of a star is the great circle PSD through the star $*$ and the pole. All the hour circles are perpendicular to the equator.

The *hour angle* of a star is the angle at the pole between the meridian and the hour circle of the star, measured from the meridian to the west. Thus the hour angle of S is $-ZPS$, negative since it is measured to the east. It is so named because, if the angle ZPS is 15° , one hour will elapse before PS coincides with PZ ; for $15^\circ = 360^\circ \div 24$, and the star appears to make a complete revolution about P in 24 hours of sidereal (i.e. star) time.

The *declination* DS of a star is its angular distance from the equator, measured on its hour circle, and positive when the star is north of the equator.

The *right ascension* of a star is the angular distance along the equator from a certain point on the equator, called the *vernal equinox*, to the foot of the hour circle through the star, measured towards the east; or it is the angle at the pole between the hour circle of the vernal equinox and that of the star.

Hence the angle between the hour circles of two stars is equal to the difference between their right ascensions.

181. At a Place in Latitude 42° N. the Altitude of a Star, whose Declination is $+60^\circ$, was measured and found to be 50° , the Star being East of the Meridian. At what Time did the Star reach the Meridian? — In the triangle ZPS , $ZP = 48^\circ$, $ZS = 40^\circ$, $PS = 30^\circ$; $\therefore$ by Art. 148, $\frac{1}{2}ZPS = 29^\circ 55'.9$; $\therefore ZPS = 59^\circ 51'.8$ or $3^h 59^m.5$. Hence the star reached the meridian $3^h 59^m.5$ after the observation was made.†

* That is, whose plane passes through the star. † Sidereal time.

182. The Latitude of the Place being 42° N., find the Interval of Time between the Rising of a Star above the Horizon and its Passage across the Meridian, its Declination being $+10^\circ$. — In the triangle ZPS , S will be on the horizon NEH at the instant of rising, so that $ZS = 90^\circ$.

$$\therefore \cos ZS = 0 = \cos ZP \cos SP + \sin ZP \sin SP \cos ZPS.$$

$$\therefore \cos ZPS = -\cot ZP \cot SP = -\cot 48^\circ \cot 80^\circ.$$

$$\therefore ZPS = 99^\circ 8'.1 \text{ or } 6^h 36^m.5.*$$

Hence the star will be above the horizon $13^h 13^m.0$.*

183. The Latitude of the Place being 42° N., and the Declination of the Star $+20^\circ$, find the Interval between the Instant when it is due East and that when it is due West. — In the triangle ZPS , $PZS = 90^\circ$.

$$\therefore \cos ZPS = \tan ZP \cot SP = \tan 48^\circ \cot 70^\circ.$$

$$\therefore ZPS = 66^\circ 9'.4. \therefore 2 ZPS = 132^\circ 18'.8 = 8^h 49^m.3.$$

Hence the interval required is $8^h 49^m.3$.*

184. The Latitude being 42° N. and the Declination of the Star $+80^\circ$, find the Azimuth of the Star when it is at its Greatest Western Elongation; that is, when the Star has reached

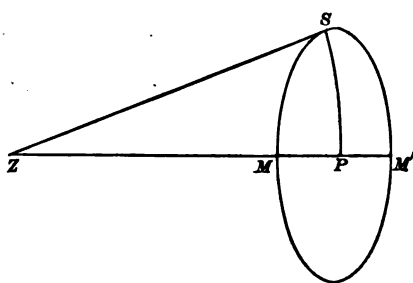


FIG. 185.

its Farthest Distance towards the West, afterwards moving East. — In the figure the ZPS triangle is projected upon the plane of the horizon, so that Z is the zenith, P the pole, S the star, MSM' the apparent diurnal path of the star about the pole, ZP the meridian, ZS the vertical circle of the star, and PZS the angle required, the angle ZSP being a right angle.

* Sidereal time.

$$\therefore \sin SP = \sin ZP \sin PZS. \quad \therefore \sin PZS = \sin 10^\circ \operatorname{cosec} 48^\circ;$$

$$\therefore PZS = 13^\circ 30'.8.$$

NOTE. — This is the method ordinarily used by the engineer to determine the north and south line.

185. The Right Ascensions of Two Stars are α and α' , and their Declinations δ and δ' ; find the Angular Distance between the Two Stars.

$$SP = 90^\circ - \delta, \quad S'P = 90^\circ - \delta', \quad SPS' = \alpha' - \alpha.$$

Hence we know two sides and the included angle, and we find the third side SS' by Art. 151 or by Art. 150.

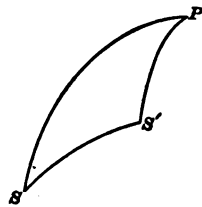


FIG. 136.

186. If α' and α'' are the Right Ascensions, and δ' and δ'' the Declinations of Two Stars, find the Inclination to the Equator of the Great Circle passing through the Stars, and also the Right Ascension of the Point where it cuts the Equator. —

Let B and D be the two stars, EQ the equator, V the vernal equinox, E the intersection of the great circle BD with the equator, $VE = \alpha_1$. In the right triangle EAB ,

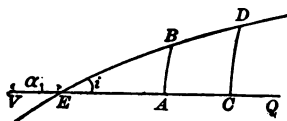


FIG. 137.

$$\sin EA = \tan AB \cot AEB. \quad \therefore \cot i = \sin(\alpha' - \alpha_1) \cot \delta'. \quad (1)$$

In the right triangle ECD ,

$$\sin EC = \tan CD \cot CED. \quad \therefore \cot i = \sin(\alpha'' - \alpha_1) \cot \delta''. \quad (2)$$

$$\therefore \frac{\sin(\alpha'' - \alpha_1)}{\sin(\alpha' - \alpha_1)} = \frac{\cot \delta'}{\cot \delta''}.$$

$$\therefore \frac{\sin(\alpha'' - \alpha_1) + \sin(\alpha' - \alpha_1)}{\sin(\alpha'' - \alpha_1) - \sin(\alpha' - \alpha_1)} = \frac{\cot \delta' + \cot \delta''}{\cot \delta' - \cot \delta''}.$$

$$\therefore \frac{\tan \frac{1}{2}(\alpha'' + \alpha' - 2\alpha_1)}{\tan \frac{1}{2}(\alpha'' - \alpha')} = \frac{\sin(\delta'' + \delta')}{\sin(\delta'' - \delta')}.$$

$$\therefore \tan \frac{1}{2}(\alpha'' + \alpha' - 2\alpha_1) = \frac{\sin(\delta'' + \delta')}{\sin(\delta'' - \delta')} \tan \frac{1}{2}(\alpha'' - \alpha'). \quad (3)$$

From (3) find $\frac{1}{2}(\alpha'' + \alpha' - 2\alpha_1)$, thence finding α_1 ; then i may be found from either (1) or (2).

187. The Right Ascension and Declination of a Star are α and δ , and those of Another Star are α' and δ' ; find the Hour Angle of the First Star and their Common Azimuth when the Stars are in the Same Vertical Circle, the Latitude of the Place being ϕ . — There are two positions, one when both stars are west, and the other when they are both east, of the meridian.

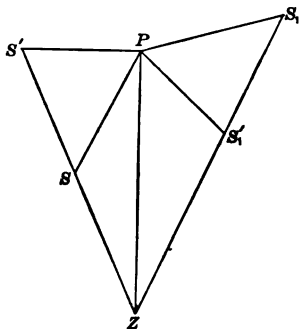


FIG. 138.

(1) $S'P = 90^\circ - \delta'$; $SP = 90^\circ - \delta$; $SPS' = \alpha - \alpha'$; $ZP = 90^\circ - \phi$. In the triangle SPS' , find $PS'S$ and PSS' . Then in the triangle $S'PZ$ we know $S'P$, ZP , and $PS'Z$, and we find PZS' and ZPS' . In the triangle SPZ we know SP , ZP , and $PSZ =$

$180^\circ - PSS'$, and we find PZS and ZPS .

The checks are $PZS' = PZS$, and $S'PZ - SPZ = \alpha - \alpha'$.

(2) $S_1P = 90^\circ - \delta$, $S'_1P = 90^\circ - \delta'$, $S_1PS'_1 = \alpha - \alpha'$; find $PS_1S'_1$ and PS'_1S_1 , these angles being the same as those at S and S' in the first case. Then from the two triangles PS_1Z and PS'_1Z we find the angles PZS_1 and PZS'_1 , which should be identical, and also the angles S_1PZ and S'_1PZ , whose difference should be $\alpha - \alpha'$.

LOGARITHMIC AND TRIGONOMETRIC TABLES

FIVE DECIMAL PLACES

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NOTE.—The well-known tables of Gauss, Becker, and Albrecht have been taken as the standards, the proof sheets have been read with great care, and it is believed that the number of errors cannot be large. The arrangement of the figures on the page is in accordance with that adopted in the standard six and seven place tables.

The natural tables were reduced from seven-place tables and compared with published five-place tables.

For convenience in using the tables, the explanation has been placed after them instead of before them.

I.

COMMON

LOGARITHMS OF NUMBERS

FROM 1 TO 11000.

N.	Log.	N.	Log.	N.	Log.	N.	Log.	N.	Log.
0	—	20	1.30 103	40	1.60 206	60	1.77 815	80	1.90 309
1	0.00 000	21	1.32 222	41	1.61 278	61	1.78 533	81	1.90 849
2	0.30 103	22	1.34 242	42	1.62 325	62	1.79 239	82	1.91 381
3	0.47 712	23	1.36 173	43	1.63 347	63	1.79 934	83	1.91 908
4	0.60 206	24	1.38 021	44	1.64 345	64	1.80 618	84	1.92 428
5	0.69 897	25	1.39 794	45	1.65 321	65	1.81 291	85	1.92 942
6	0.77 815	26	1.41 497	46	1.66 276	66	1.81 954	86	1.93 450
7	0.84 510	27	1.43 136	47	1.67 210	67	1.82 607	87	1.93 952
8	0.90 309	28	1.44 716	48	1.68 124	68	1.83 251	88	1.94 448
9	0.95 424	29	1.46 240	49	1.69 020	69	1.83 885	89	1.94 939
10	1.00 000	30	1.47 712	50	1.69 897	70	1.84 510	90	1.95 424
11	1.04 139	31	1.49 136	51	1.70 757	71	1.85 126	91	1.95 904
12	1.07 918	32	1.50 515	52	1.71 600	72	1.85 733	92	1.96 379
13	1.11 394	33	1.51 851	53	1.72 428	73	1.86 332	93	1.96 848
14	1.14 613	34	1.53 148	54	1.73 239	74	1.86 923	94	1.97 313
15	1.17 609	35	1.54 407	55	1.74 036	75	1.87 506	95	1.97 772
16	1.20 412	36	1.55 630	56	1.74 819	76	1.88 081	96	1.98 227
	1.23 045	37	1.56 820	57	1.75 587	77	1.88 649	97	1.98 677
	1.25 527	38	1.57 978	58	1.76 343	78	1.89 209	98	1.99 123
	1.27 875	39	1.59 106	59	1.77 085	79	1.89 763	99	1.99 564
		40	1.60 206	60	1.77 815	80	1.90 309	100	2.00 000
		T'.				S''.		T''.	
		373		0° 0' = 0''		4.68 557		557	
3		373		0 1 = 60		557		557	
				0 2 = 120		557		557	

S'. T'.	N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
6.46 366 385	100	00 000	043	087	130	173	217	260	303	346	389	44 43 42
366 385	101	432	475	518	561	604	647	689	732	775	817	1 4.4 4.3 4.2
366 385	102	860	903	945	988	*030	*072	*115	*157	*199	*242	2 8.8 8.6 8.4
366 386	103	01 284	326	368	410	452	494	536	578	620	662	3 13.2 12.9 12.6
366 386	104	703	745	787	828	870	912	953	995	*036	*078	4 17.6 17.2 16.8
366 386	105	02 119	160	202	243	284	325	366	407	449	490	5 22.0 21.5 21.0
366 386	106	531	572	612	653	694	735	776	816	857	898	6 26.4 25.8 25.2
366 387	107	938	979	*019	*060	*100	*141	*181	*222	*262	*302	7 30.8 30.1 29.4
365 387	108	03 342	383	423	463	503	543	583	623	663	703	8 35.2 34.4 33.6
365 387	109	743	782	822	862	902	941	981	*021	*060	*100	9 39.6 38.7 37.8
365 387	110	04 139	179	218	258	297	336	376	415	454	493	41 40 39
365 388	111	532	571	610	650	689	727	766	805	844	883	1 4.1 4.0 3.9
365 388	112	922	961	999	*038	*077	*115	*154	*192	*231	*269	2 8.2 8.0 7.8
365 388	113	05 308	346	385	423	461	500	538	576	614	652	3 12.3 12.0 11.7
365 389	114	690	729	767	805	843	881	918	956	994	*032	4 16.4 16.0 15.6
365 389	115	06 070	108	145	183	221	258	296	333	371	408	5 20.5 20.0 19.5
364 389	116	446	483	521	558	595	633	670	707	744	781	6 24.6 24.0 23.4
364 389	117	819	856	893	930	967	*004	*041	*078	*115	*151	7 28.7 28.0 27.3
364 390	118	07 188	225	262	298	335	372	408	445	482	518	8 32.8 32.0 31.2
364 390	119	555	591	628	664	700	737	773	809	846	882	9 36.9 36.0 35.1
364 391	120	918	954	990	*027	*063	*099	*135	*171	*207	*243	38 37 36
364 391	121	08 279	314	350	386	422	458	493	529	565	600	1 3.8 3.7 3.6
363 391	122	636	672	707	743	778	814	849	884	920	955	2 7.6 7.4 7.2
363 391	123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307	3 11.4 11.1 10.8
363 392	124	09 342	377	412	447	482	517	552	587	621	656	4 15.2 14.8 14.4
363 392	125	691	726	760	795	830	864	899	934	968	*003	5 19.0 18.5 18.0
363 392	126	10 037	072	106	140	175	209	243	278	312	346	6 22.8 22.2 21.6
363 392	127	380	415	449	483	517	551	585	619	653	687	7 26.6 25.9 25.2
363 393	128	721	755	789	823	857	890	924	958	992	*025	8 30.4 29.6 28.8
362 393	129	11 059	093	126	160	193	227	261	294	327	361	9 34.2 33.3 32.4
362 393	130	394	428	461	494	528	561	594	628	661	694	35 34 33
362 394	131	727	760	793	826	860	893	926	959	992	*024	1 3.5 3.4 3.3
362 394	132	12 057	090	123	156	189	222	254	287	320	352	2 7.0 6.8 6.6
362 394	133	385	418	450	483	516	548	581	613	646	678	3 10.5 10.2 9.9
362 395	134	710	743	775	808	840	872	905	937	969	*001	4 14.0 13.6 13.2
361 395	135	13 033	066	098	130	162	194	226	258	290	322	5 17.5 17.0 16.5
361 395	136	354	386	418	450	481	513	545	577	609	640	6 21.0 20.4 19.8
361 396	137	672	704	735	767	799	830	862	893	925	956	7 24.5 23.8 23.1
361 396	138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270	8 28.0 27.2 26.4
361 396	139	14 301	333	364	395	426	457	489	520	551	582	9 31.5 30.6 29.7
361 397	140	613	644	675	706	737	768	799	829	860	891	32 31 30
360 397	141	922	953	983	*014	*045	*076	*106	*137	*168	*198	1 3.2 3.1 3.0
360 397	142	15 229	259	290	320	351	381	412	442	473	503	2 6.4 6.2 6.0
360 398	143	534	564	594	625	655	685	715	746	776	806	3 9.6 9.3 9.0
360 398	144	836	866	897	927	957	987	*017	*047	*077	*107	4 12.8 12.4 12.0
360 398	145	16 137	167	197	227	256	286	316	346	376	406	5 16.0 15.5 15.0
360 399	146	435	465	495	524	554	584	613	643	673	702	6 19.2 18.6 18.0
359 399	147	732	761	791	820	850	879	909	938	967	997	7 22.4 21.7 21.0
359 399	148	17 026	056	085	114	143	173	202	231	260	289	8 25.6 24.8 24.0
359 400	149	319	348	377	406	435	464	493	522	551	580	9 28.8 27.9 27.0
359 400	150	609	638	667	696	725	754	782	811	840	869	
	N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
S' T'												S' T'
1' 6.46	373	373	0°	1' = 60''	4.68	557	557	0°	19' = 1140''	4.68	557	55
2	373	373	0	2 = 120		557	557	0	20 = 1200		557	55 ⁸
10	373	373	0	3 = 180		557	557	0	21 = 1260		557	55 ⁸
13	373	373	0	16 = 960		557	558	0	22 = 1320		557	55 ⁸
14	372	373	0	17 = 1020		557	558	0	23 = 1380		557	55 ⁸
15	372	373	0	18 = 1080		557	558	0	24 = 1440		557	55 ⁸
			0	19 = 1140		557	558	0	25 = 1500		557	55 ⁸

S'	T'	N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.	
359	400	150	17 609	638	667	696	725	754	782	811	840	869	29	28
359	401	151	898	926	955	984	*013	*041	*070	*099	*127	*156	1	2.9 2.8
358	401	152	18 184	213	241	270	298	327	355	384	412	441	2	5.3 5.6
358	401	153	469	498	526	554	583	611	639	667	696	724	3	8.7 8.4
358	402	154	752	780	808	837	865	893	921	949	977	*005	4	11.6 11.2
358	402	155	19 033	061	089	117	145	173	201	229	257	285	5	14.5 14.0
358	402	156	312	340	368	396	424	451	479	507	535	562	6	17.4 16.8
358	403	157	590	618	645	673	700	728	756	783	811	838	7	20.3 19.6
357	403	158	866	893	921	948	976	*003	*030	*058	*085	*112	8	23.2 22.4
357	404	159	20 140	167	194	222	249	276	303	330	358	385	9	26.1 25.2
357	404	160	412	439	466	493	520	548	575	602	629	656	27	26
357	404	161	683	710	737	763	790	817	844	871	898	925	1	2.7 2.6
357	405	162	952	978	*005	*032	*059	*085	*112	*139	*165	*192	2	5.4 5.2
356	405	163	21 219	245	272	299	325	352	378	405	431	458	3	8.1 7.8
356	406	164	484	511	537	564	590	617	643	669	696	722	4	10.8 10.4
356	406	165	748	775	801	827	854	880	906	932	958	985	5	13.5 13.0
356	406	166	22 011	037	063	089	115	141	167	194	220	246	6	16.2 15.6
356	407	167	272	298	324	350	376	401	427	453	479	505	7	18.9 18.2
355	407	168	531	557	583	608	634	660	686	712	737	763	8	21.6 20.8
355	408	169	789	814	840	866	891	917	943	968	994	*019	9	24.3 23.4
355	408	170	23 045	070	096	121	147	172	198	223	249	274	25	
355	408	171	300	325	350	376	401	426	452	477	502	528	1	2.5
354	409	172	553	578	603	629	654	679	704	729	754	779	2	5.0
354	409	173	805	830	855	880	905	930	955	980	*005	*030	3	7.5
354	410	174	24 055	080	105	130	155	180	204	229	254	279	4	10.0
354	410	175	304	329	353	378	403	428	452	477	502	527	5	12.5
354	411	176	551	576	601	625	650	674	699	724	748	773	6	15.0
353	411	177	797	822	846	871	895	920	944	969	993	*018	7	17.5
353	411	178	25 042	066	091	115	139	164	188	212	237	261	8	20.0
353	412	179	285	310	334	358	382	406	431	455	479	503	9	22.5
353	412	180	527	551	575	600	624	648	672	696	720	744	24	23
353	413	181	768	792	816	840	864	888	912	935	959	983	1	2.4 2.3
352	413	182	26 007	031	055	079	102	126	150	174	198	221	2	4.8 4.6
352	414	183	245	269	293	316	340	364	387	411	435	458	3	7.2 6.9
352	414	184	482	505	529	553	576	600	623	647	670	694	4	9.6 9.2
352	415	185	717	741	764	788	811	834	858	881	905	928	5	12.0 11.5
351	415	186	951	975	*021	*045	*068	*091	*114	*138	*161		6	14.4 13.8
351	415	187	27 184	207	231	254	277	300	323	346	370	393	7	16.8 16.1
351	416	188	416	439	462	485	508	531	554	577	600	623	8	19.2 18.4
351	416	189	646	669	692	715	738	761	784	807	830	852	9	21.6 20.7
350	417	190	875	898	921	944	967	989	*012	*035	*058	*081	22	21
350	417	191	28 103	126	149	171	194	217	240	262	285	307	1	2.2 2.1
350	418	192	330	353	375	398	421	443	466	488	511	533	2	4.4 4.2
350	418	193	556	578	601	623	646	668	691	713	735	758	3	6.6 6.3
350	419	194	780	803	825	847	870	892	914	937	959	981	4	8.8 8.4
349	419	195	29 003	026	048	070	092	115	137	159	181	203	5	11.0 10.5
349	420	196	226	248	270	292	314	336	358	380	403	425	6	13.2 12.6
349	420	197	447	469	491	513	535	557	579	601	623	645	7	15.4 14.7
349	421	198	667	688	710	732	754	776	798	820	842	863	8	17.6 16.8
348	421	199	885	907	929	951	973	994	*016	*038	*060	*081	9	19.8 18.9
348	422	200	30 103	125	146	168	190	211	233	255	276	298		
		N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.	
		S'	T'										S''	T''
1'	6.46	373	373										4.68	557 558
2		373	373										557	559
15		372	373										557	559
20		372	373										557	559
				o	2' = 120''	4.68	557	557	557	557	557	557	557	559
				o	3 = 180		557	557	557	557	557	557	557	559
				o	4 = 240		557	558	557	557	557	557	557	559
				o	25 = 1500		557	558	557	557	557	557	557	559
				o	26 = 1560		557	558	557	557	557	557	557	559
				o	27 = 1620		557	558	557	557	557	557	557	559
				o	28 = 1680		557	558	557	557	557	557	557	559

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.
200	30	103	125	146	168	190	211	233	255	276	298	22 21
201		320	341	363	384	406	428	449	471	492	514	1 2.2 2.1
202		535	557	578	600	621	643	664	685	707	728	2 4.4 4.2
203		750	771	792	814	835	856	878	899	920	942	3 6.6 6.3
204		963	984	*006	*027	*048	*069	*091	*112	*133	*154	4 8.8 8.4
205	31	175	197	218	239	260	281	302	323	345	366	5 11.0 10.5
206		387	408	429	450	471	492	513	534	555	576	6 13.2 12.6
207		597	618	639	660	681	702	723	744	765	785	7 15.4 14.7
208		806	827	848	869	890	911	931	952	973	994	8 17.6 16.8
209	32	015	035	056	077	098	118	139	160	181	201	9 19.8 18.9
210		222	243	263	284	305	325	346	366	387	408	20
211		428	449	469	490	510	531	552	572	593	613	1 2.0
212		634	654	675	695	715	736	756	777	797	818	2 4.0
213		838	858	879	899	919	940	960	980	*001	*021	3 6.0
214	33	041	062	082	102	122	143	163	183	203	224	4 8.0
215		244	264	284	304	325	345	365	385	405	425	5 10.0
216		445	465	486	506	526	546	566	586	606	626	6 12.0
217		646	666	686	706	726	746	766	786	806	826	7 14.0
218		846	866	885	905	925	945	965	985	*005	*025	8 16.0
219	34	044	064	084	104	124	143	163	183	203	223	9 18.0
220		242	262	282	301	321	341	361	380	400	420	19
221		439	459	479	498	518	537	557	577	596	616	1 1.9
222		635	655	674	694	713	733	753	772	792	811	2 3.8
223		830	850	869	889	908	928	947	967	986	*005	3 5.7
224	35	025	044	064	083	102	122	141	160	180	199	4 7.6
225		218	238	257	276	295	315	334	353	372	392	5 9.5
226		411	430	449	468	488	507	526	545	564	583	6 11.4
227		603	622	641	660	679	698	717	736	755	774	7 13.3
228		793	813	832	851	870	889	908	927	946	965	8 15.2
229		984	*003	*021	*040	*059	*078	*097	*116	*135	*154	9 17.1
230	36	173	192	211	229	248	267	286	305	324	342	18
231		361	380	399	418	436	455	474	493	511	530	1 1.8
232		549	568	586	605	624	642	661	680	698	717	2 3.6
233		736	754	773	791	810	829	847	866	884	903	3 5.4
234		922	940	959	977	996	*014	*033	*051	*070	*088	4 7.2
235	37	107	125	144	162	181	199	218	236	254	273	5 9.0
236		291	310	328	346	365	383	401	420	438	457	6 10.8
237		475	493	511	530	548	566	585	603	621	639	7 12.6
238		658	676	694	712	731	749	767	785	803	822	8 14.4
239		840	858	876	894	912	931	949	967	985	*003	9 16.2
240	38	021	039	057	075	093	112	130	148	166	184	17
241		202	220	238	256	274	292	310	328	346	364	1 1.7
242		382	399	417	435	453	471	489	507	525	543	2 3.4
243		561	578	596	614	632	650	668	686	703	721	3 5.1
244		739	757	775	792	810	828	846	863	881	899	4 6.8
245		917	934	952	970	987	*005	*023	*041	*058	*076	5 8.5
246	39	094	111	129	146	164	182	199	217	235	252	6 10.2
247		270	287	305	322	340	358	375	393	410	428	7 11.9
248		445	463	480	498	515	533	550	568	585	602	8 13.6
249		620	637	655	672	690	707	724	742	759	777	9 15.3
250		794	811	829	846	863	881	898	915	933	950	

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.	
		S.'	T.'				S."	T."				S."	T."
2'	6.46	373	373	0°	3' = 180"	4.68	557	557	0°	36' = 2160"	4.68	557	557
3		373	373	0	4 = 240		557	558	0	37 = 2220		557	557
				0	5 = 300		557	558	0	38 = 2280		557	557
20		372	373						0	39 = 2340		557	557
25		372	373	0	33 = 1980		557	559	0	40 = 2400		557	557
				0	34 = 2040		557	559	0	41 = 2460		556	556
				0	35 = 2100		557	559	0	42 = 2520		556	556
				0	36 = 2160		557	559					

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.	
250	39	794	811	829	846	863	881	898	915	933	950	18	
251		967	985	*002	*019	*037	*054	*071	*088	*106	*123	1	1.8
252	40	140	157	175	192	209	226	243	261	278	295	2	3.6
253		312	329	346	364	381	398	415	432	449	466	3	5.4
254		483	500	518	535	552	569	586	603	620	637	4	7.2
255		654	671	688	705	722	739	756	773	790	807	5	9.0
256		824	841	858	875	892	909	926	943	960	976	6	10.8
257		993	*010	*027	*044	*061	*078	*095	*111	*128	*145	7	12.6
258	41	162	179	196	212	229	246	263	280	296	313	8	14.4
259		330	347	363	380	397	414	430	447	464	481	9	16.2
260		497	514	531	547	564	581	597	614	631	647	17	
261		664	681	697	714	731	747	764	780	797	814	1	1.7
262		830	847	863	880	896	913	929	946	963	979	2	3.4
263		996	*012	*029	*045	*062	*078	*095	*111	*127	*144	3	5.1
264	42	160	177	193	210	226	243	259	275	292	308	4	6.8
265		325	341	357	374	390	406	423	439	455	472	5	8.5
266		488	504	521	537	553	570	586	602	619	635	6	10.2
267		651	667	684	700	716	732	749	765	781	797	7	11.9
268		813	830	846	862	878	894	911	927	943	959	8	13.6
269		975	991	*008	*024	*040	*056	*072	*088	*104	*120	9	15.3
270	43	136	152	169	185	201	217	233	249	265	281	16	
271		297	313	329	345	361	377	393	409	425	441	1	1.6
272		457	473	489	505	521	537	553	569	584	600	2	3.2
273		616	632	648	664	680	696	712	727	743	759	3	4.8
274		775	791	807	823	838	854	870	886	902	917	4	6.4
275		933	949	965	981	996	*012	*028	*044	*059	*075	5	8.0
276	44	091	107	122	138	154	170	185	201	217	232	6	9.6
277		248	264	279	295	311	326	342	358	373	389	7	11.2
278		404	420	436	451	467	483	498	514	529	545	8	12.8
279		560	576	592	607	623	638	654	669	685	700	9	14.4
280		716	731	747	762	778	793	809	824	840	855	15	
281		871	886	902	917	932	948	963	979	994	*010	1	1.5
282	45	025	040	056	071	086	102	117	133	148	163	2	3.0
283		179	194	209	225	240	255	271	286	301	317	3	4.5
284		332	347	362	378	393	408	423	439	454	469	4	6.0
285		484	500	515	530	545	561	576	591	606	621	5	7.5
286		637	652	667	682	697	712	728	743	758	773	6	9.0
287		788	803	818	834	849	864	879	894	909	924	7	10.5
288		939	954	969	984	*000	*015	*030	*045	*060	*075	8	12.0
289	46	090	105	120	135	150	165	180	195	210	225	9	13.5
290		240	255	270	285	300	315	330	345	359	374	14	
291		389	404	419	434	449	464	479	494	509	523	1	1.4
292		538	553	568	583	598	613	627	642	657	672	2	2.8
293		687	702	716	731	746	761	776	790	805	820	3	4.2
294		835	850	864	879	894	909	923	938	953	967	4	5.6
295		982	997	*012	*026	*041	*056	*070	*085	*100	*114	5	7.0
296	47	129	144	159	173	188	202	217	232	246	261	6	8.4
297		276	290	305	319	334	349	363	378	392	407	7	9.8
298		422	436	451	465	480	494	509	524	538	553	8	11.2
299		567	582	596	611	625	640	654	669	683	698	9	12.6
300		712	727	741	756	770	784	799	813	828	842		

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.	
	S.	T.					S.	T.				S.	T.
373	373		0° 4' = 240"	4.68	557	558	0° 45' = 2700"	4.68	556	560			
373	373		0 5 = 300		557	558	0 46 = 2760		556	560			
372	373		0 41 = 2460		556	560	0 47 = 2820		556	560			
372	373		0 42 = 2520		556	560	0 48 = 2880		556	560			
372	374		0 43 = 2580		556	560	0 49 = 2940		556	560			
372	374		0 44 = 2640		556	560	0 50 = 3000		556	561			
			0 45 = 2700		556	560							

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.		
300	47	712	727	741	756	770	784	799	813	828	842			
301		857	871	885	900	914	929	943	958	972	986			
302	48	001	015	029	044	058	073	087	101	116	130			
303		144	159	173	187	202	216	230	244	259	273	15		
304		287	302	316	330	344	359	373	387	401	416	1 1.5		
305		430	444	458	473	487	501	515	530	544	558	2 3.0		
306		572	586	601	615	629	643	657	671	686	700	3 4.5		
307		714	728	742	756	770	785	799	813	827	841	4 6.0		
308		855	869	883	897	911	926	940	954	968	982	5 7.5		
309		996	*010	*024	*038	*052	*066	*080	*094	*108	*122	6 9.0		
310	49	136	150	164	178	192	206	220	234	248	262	7 10.5		
311		276	290	304	318	332	346	360	374	388	402	8 12.0		
312		415	429	443	457	471	485	499	513	527	541	9 13.5		
313		554	568	582	596	610	624	638	651	665	679			
314		693	707	721	734	748	762	776	790	803	817	14		
315		831	845	859	872	886	900	914	927	941	955	1 1.4		
316		969	982	996	*010	*024	*037	*051	*065	*079	*092	2 2.8		
317	50	106	120	133	147	161	174	188	202	215	229	3 4.2		
318		243	256	270	284	297	311	325	338	352	365	4 5.6		
319		379	393	406	420	433	447	461	474	488	501	5 7.0		
320		515	529	542	556	569	583	596	610	623	637	6 8.4		
321		651	664	678	691	705	718	732	745	759	772	7 9.8		
322		786	799	813	826	840	853	866	880	893	907	8 11.2		
323		920	934	947	961	974	987	*001	*014	*028	*041	9 12.6		
324	51	055	068	081	095	108	121	135	148	162	175			
325		188	202	215	228	242	255	268	282	295	308			
326		322	335	348	362	375	388	402	415	428	441	18		
327		455	468	481	495	508	521	534	548	561	574	1 1.3		
328		587	601	614	627	640	654	667	680	693	706	2 2.6		
329		720	733	746	759	772	786	799	812	825	838	3 3.9		
330		851	865	878	891	904	917	930	943	957	970	4 5.2		
331		983	996	*009	*022	*035	*048	*061	*075	*088	*101	5 6.5		
332	52	114	127	140	153	166	179	192	205	218	231	6 7.8		
333		244	257	270	284	297	310	323	336	349	362	7 9.1		
334		375	388	401	414	427	440	453	466	479	492	8 10.4		
335		504	517	530	543	556	569	582	595	608	621	9 11.7		
336		634	647	660	673	686	699	711	724	737	750			
337		763	776	789	802	815	827	840	853	866	879			
338		892	905	917	930	943	956	969	982	994	*007			
339	53	020	033	046	058	071	084	097	110	122	135	18		
340		148	161	173	186	199	212	224	237	250	263	1 1.2		
341		275	288	301	314	326	339	352	364	377	390	2 2.4		
342		403	415	428	441	453	466	479	491	504	517	3 3.6		
343		529	542	555	567	580	593	605	618	631	643	4 4.8		
344		656	668	681	694	706	719	732	744	757	769	5 6.0		
345		782	794	807	820	832	845	857	870	882	895	6 7.2		
346		908	920	933	945	958	970	983	995	*008	*020	7 8.4		
347	54	033	045	058	070	083	095	108	120	133	145	8 9.6		
348		158	170	183	195	208	220	233	245	258	270	9 10.8		
349		283	295	307	320	332	345	357	370	382	394			
350		407	419	432	444	456	469	481	494	506	518			
N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.		
		S.' T.'				S.' T.'					S.' T.			
3'	6.46	373	373			0° 5' = 300"	4.68	557	558		0° 54' = 3240"	4.68	556	56
4		373	373			0 6 = 360		557	558		0 55 = 3300		556	56
30		372	374			0 50 = 3000		556	561		0 56 = 3360		556	56
35		372	374			0 51 = 3060		556	561		0 57 = 3420		555	56
						0 52 = 3120		556	561		0 58 = 3480		555	56
						0 53 = 3180		556	561		0 59 = 3540		555	56
						0 54 = 3240		556	561					

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.				
350	54	407	419	432	444	456	469	481	494	506	518					
351		531	543	555	568	580	593	605	617	630	642					
352		654	667	679	691	704	716	728	741	753	765	18				
353		777	790	802	814	827	839	851	864	876	888	1				
354		900	913	925	937	949	962	974	986	998	*011	2				
355	55	023	035	047	060	072	084	096	108	121	133	3				
356		145	157	169	182	194	206	218	230	242	255	4				
357		267	279	291	303	315	328	340	352	364	376	5				
358		388	400	413	425	437	449	461	473	485	497	6				
359		509	522	534	546	558	570	582	594	606	618	7				
360		630	642	654	666	678	691	703	715	727	739	8				
361		751	763	775	787	799	811	823	835	847	859	9				
362		871	883	895	907	919	931	943	955	967	979	11.7				
363		991	*003	*015	*027	*038	*050	*062	*074	*086	*098					
364	56	110	122	134	146	158	170	182	194	205	217	12				
365		229	241	253	265	277	289	301	312	324	336					
366		348	360	372	384	396	407	419	431	443	455	1				
367		467	478	490	502	514	526	538	549	561	573	2				
368		585	597	608	620	632	644	656	667	679	691	3				
369		703	714	726	738	750	761	773	785	797	808	4				
370		820	832	844	855	867	879	891	902	914	926	5				
371		937	949	961	972	984	996	*008	*019	*031	*043	6				
372	57	054	066	078	089	101	113	124	136	148	159	7				
373		171	183	194	206	217	229	241	252	264	276	8				
374		287	299	310	322	334	345	357	368	380	392	9				
375		403	415	426	438	449	461	473	484	496	507	10.8				
376		519	530	542	553	565	576	588	600	611	623					
377		634	646	657	669	680	692	703	715	726	738	11				
378		749	761	772	784	795	807	818	830	841	852	1				
379		864	875	887	898	910	921	933	944	955	967	2				
380		978	990	*001	*013	*024	*035	*047	*058	*070	*081	3				
381	58	092	104	115	127	138	149	161	172	184	195	4				
382		206	218	229	240	252	263	274	286	297	309	5				
383		320	331	343	354	365	377	388	399	410	422	6				
384		433	444	456	467	478	490	501	512	524	535	7				
385		546	557	569	580	591	602	614	625	636	647	8				
386		659	670	681	692	704	715	726	737	749	760	9				
387		771	782	794	805	816	827	838	850	861	872					
388		883	894	906	917	928	939	950	961	973	984					
389		995	*006	*017	*028	*040	*051	*062	*073	*084	*095	10				
390	59	106	118	129	140	151	162	173	184	195	207	1				
391		218	229	240	251	262	273	284	295	306	318	2				
392		329	340	351	362	373	384	395	406	417	428	3				
393		439	450	461	472	483	494	506	517	528	539	4				
394		550	561	572	583	594	605	616	627	638	649	5				
395		660	671	682	693	704	715	726	737	748	759	6				
396		770	780	791	802	813	824	835	846	857	868	7				
397		879	890	901	912	923	934	945	956	966	977	8				
398		988	999	*010	*021	*032	*043	*054	*065	*076	*086	9				
399	60	097	108	119	130	141	152	163	173	184	195					
400		206	217	228	239	249	260	271	282	293	304					
N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.				
		S. /	T. /				S. /	T. /				S. /	T. /			
		6.46	373	373	0°	5' =	300"	4.68	557	558	1°	1' =	3660"	4.68	555	562
			373	373	0	6 =	360		557	558	1	2 =	3720		555	562
			372	374	0	7 =	420		557	558	1	3 =	3780		555	562
			372	374	0	8 =	3480		555	562	1	4 =	3840		555	563
			372	375	0	59 =	3540		555	562	1	5 =	3900		555	563
					1	0 =	3600		555	562	1	6 =	3960		555	563
					1	1 =	3660		555	562	1	7 =	4020		555	563

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.
00	60	206	217	228	239	249	260	271	282	293	304	
101		314	325	336	347	358	369	379	390	401	412	
102		423	433	444	455	466	477	487	498	509	520	
103		531	541	552	563	574	584	595	606	617	627	
104		638	649	660	670	681	692	703	713	724	735	
105		746	756	767	778	788	799	810	821	831	842	
106		853	863	874	885	895	906	917	927	938	949	
107		959	970	981	991	*002	*013	*023	*034	*045	*055	11
108	61	066	077	087	098	109	119	130	140	151	162	1 1.1
109		172	183	194	204	215	225	236	247	257	268	2 2.2
110		278	289	300	310	321	331	342	352	363	374	3 3.3
111		384	395	405	416	426	437	448	458	469	479	4 4.4
112		490	500	511	521	532	542	553	563	574	584	5 5.5
113		595	606	616	627	637	648	658	669	679	690	6 6.6
114		700	711	721	731	742	752	763	773	784	794	7 7.7
115		805	815	826	836	847	857	868	878	888	899	8 8.8
116		909	920	930	941	951	962	972	982	993	*003	9 9.9
117	62	014	024	034	045	055	066	076	086	097	107	
118		118	128	138	149	159	170	180	190	201	211	
119		221	232	242	252	263	273	284	294	304	315	
120		325	335	346	356	366	377	387	397	408	418	10
121		428	439	449	459	469	480	490	500	511	521	1 1.0
122		531	542	552	562	572	583	593	603	613	624	2 2.0
123		634	644	655	665	675	685	696	706	716	726	3 3.0
124		737	747	757	767	778	788	798	808	818	829	4 4.0
125		839	849	859	870	880	890	900	910	921	931	5 5.0
126		941	951	961	972	982	992	*002	*012	*022	*033	6 6.0
127	63	043	053	063	073	083	094	104	114	124	134	7 7.0
128		144	155	165	175	185	195	205	215	225	236	8 8.0
129		246	256	266	276	286	296	306	317	327	337	9 9.0
130		347	357	367	377	387	397	407	417	428	438	
131		448	458	468	478	488	498	508	518	528	538	
132		548	558	568	579	589	599	609	619	629	639	
133		649	659	669	679	689	699	709	719	729	739	
134		749	759	769	779	789	799	809	819	829	839	
135		849	859	869	879	889	899	909	919	929	939	
136		949	959	969	979	988	998	*008	*018	*028	*038	
137	64	048	058	068	078	088	098	108	118	128	137	9
138		147	157	167	177	187	197	207	217	227	237	1 0.9
139		246	256	266	276	286	296	306	316	326	335	2 1.8
140		345	355	365	375	385	395	404	414	424	434	3 2.7
141		444	454	464	473	483	493	503	513	523	532	4 3.6
142		542	552	562	572	582	591	601	611	621	631	5 4.5
143		640	650	660	670	680	689	699	709	719	729	6 5.4
144		738	748	758	768	777	787	797	807	816	826	7 6.3
145		836	846	856	865	875	885	895	904	914	924	8 7.2
146		933	943	953	963	972	982	992	*002	*011	*021	9 8.1
147	65	031	040	050	060	070	079	089	099	108	118	
148		128	137	147	157	167	176	186	196	205	215	
149		225	234	244	254	263	273	283	292	302	312	
150		321	331	341	350	360	369	379	389	398	408	

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.
		S.	T.				S.	T.			S.	T.
4'	6.46	373	373		0° 6' = 360"	4.68	557	558	1° 9' = 4140"	4.68	553	56
5		373	373		0 7 = 420		557	558	1 10 = 4200		554	56
40		372	375		0 8 = 480		557	558	1 11 = 4260		554	56
42		372	375		1 6 = 3960		555	563	1 12 = 4320		554	56
43		371	375		1 7 = 4020		555	563	1 13 = 4380		554	56
44		371	375		1 8 = 4080		555	563	1 14 = 4440		554	56
45		371	375		1 9 = 4140		555	563	1 15 = 4500		554	56

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.			
480	65	321	331	341	350	360	369	379	389	398	408				
451		418	427	437	447	456	466	475	485	495	504				
452		514	523	533	543	552	562	571	581	591	600				
453		610	619	629	639	648	658	667	677	686	696				
454		706	715	725	734	744	753	763	772	782	792				
455		801	811	820	830	839	849	858	868	877	887				
456		896	906	916	925	935	944	954	963	973	982	10			
457		992	*001	*011	*020	*030	*039	*049	*058	*068	*077	1 1.0			
458	66	087	096	106	115	124	134	143	153	162	172	2 2.0			
459		181	191	200	210	219	229	238	247	257	266	3 3.0			
460		276	285	295	304	314	323	332	342	351	361	4 4.0			
461		370	380	389	398	408	417	427	436	445	455	5 5.0			
462		464	474	483	492	502	511	521	530	539	549	6 6.0			
463		558	567	577	586	596	605	614	624	633	642	7 7.0			
464		652	661	671	680	689	699	708	717	727	736	8 8.0			
465		745	755	764	773	783	792	801	811	820	829	9 9.0			
466		839	848	857	867	876	885	894	904	913	922				
467		932	941	950	960	969	978	987	997	*006	*015				
468	67	025	034	043	052	062	071	080	089	099	108				
469		117	127	136	145	154	164	173	182	191	201				
470		210	219	228	237	247	256	265	274	284	293	9			
471		302	311	321	330	339	348	357	367	376	385	1 0.9			
472		394	403	413	422	431	440	449	459	468	477	2 1.8			
473		486	495	504	514	523	532	541	550	560	569	3 2.7			
474		578	587	596	605	614	624	633	642	651	660	4 3.6			
475		669	679	688	697	706	715	724	733	742	752	5 4.5			
476		761	770	779	788	797	806	815	825	834	843	6 5.4			
477		852	861	870	879	888	897	906	916	925	934	7 6.3			
478		943	952	961	970	979	988	997	*006	*015	*024	8 7.2			
479	68	034	043	052	061	070	079	088	097	106	115	9 8.1			
480		124	133	142	151	160	169	178	187	196	205				
481		215	224	233	242	251	260	269	278	287	296				
482		305	314	323	332	341	350	359	368	377	386				
483		395	404	413	422	431	440	449	458	467	476				
484		485	494	502	511	520	529	538	547	556	565				
485		574	583	592	601	610	619	628	637	646	655				
486		664	673	681	690	699	708	717	726	735	744				
487		753	762	771	780	789	797	806	815	824	833	8			
488		842	851	860	869	878	886	895	904	913	922	1 0.8			
489		931	940	949	958	966	975	984	993	*002	*011	2 1.6			
490	69	020	028	037	046	055	064	073	082	090	099	3 2.4			
491		108	117	126	135	144	152	161	170	179	188	4 3.2			
492		197	205	214	223	232	241	249	258	267	276	5 4.0			
493		285	294	302	311	320	329	338	346	355	364	6 4.8			
494		373	381	390	399	408	417	425	434	443	452	7 5.6			
495		461	469	478	487	496	504	513	522	531	539	8 6.4			
496		548	557	566	574	583	592	601	609	618	627	9 7.2			
497		636	644	653	662	671	679	688	697	705	714				
498		723	732	740	749	758	767	775	784	793	801				
499		810	819	827	836	845	854	862	871	880	888				
500		897	906	914	923	932	940	949	958	966	975				
L.	U	1	2	3	4	5	6	7	8	9	P. P.				
S.'	T.'					S."	T."					S."	T."		
373	373	0° 7' = 420"				4.68	557	558	1° 18' = 4680"				4.68	554	565
373	373	0 8 = 480					557	558	1 19 = 4740					554	565
		0 9 = 540					557	558	1 20 = 4800					554	565
		1 15 = 4500					554	564	1 21 = 4860					553	566
		1 16 = 4560					554	565	1 22 = 4920					553	566
		1 17 = 4620					554	565	1 23 = 4980					553	566
371	376	1 18 = 4680					554	565	1 24 = 5040					553	566

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.
500	69	897	906	914	923	932	940	949	958	966	975	
501		984	992	*001	*010	*018	*027	*036	*044	*053	*062	
502	70	070	079	088	096	105	114	122	131	140	148	
503		157	165	174	183	191	200	209	217	226	234	
504		243	252	260	269	278	286	295	303	312	321	
505		329	338	346	355	364	372	381	389	398	406	
506		415	424	432	441	449	458	467	475	484	492	9
507		501	509	518	526	535	544	552	561	569	578	1 0.9
508		586	595	603	612	621	629	638	646	655	663	2 1.8
509		672	680	689	697	706	714	723	731	740	749	3 2.7
510		757	766	774	783	791	800	808	817	825	834	4 3.6
511		842	851	859	868	876	885	893	902	910	919	5 4.5
512		927	935	944	952	961	969	978	986	995	*003	6 5.4
513	71	012	020	029	037	046	054	063	071	079	088	7 6.3
514		096	105	113	122	130	139	147	155	164	172	8 7.2
515		181	189	198	206	214	223	231	240	248	257	9 8.1
516		265	273	282	290	299	307	315	324	332	341	
517		349	357	366	374	383	391	399	408	416	425	
518		433	441	450	458	466	475	483	492	500	508	
519		517	525	533	542	550	559	567	575	584	592	
520		600	609	617	625	634	642	650	659	667	675	8
521		684	692	700	709	717	725	734	742	750	759	1 0.8
522		767	775	784	792	800	809	817	825	834	842	2 1.6
523		850	858	867	875	883	892	900	908	917	925	3 2.4
524		933	941	950	958	966	975	983	991	999	*008	4 3.2
525	72	016	024	032	041	049	057	066	074	082	090	5 4.0
526		099	107	115	123	132	140	148	156	165	173	6 4.8
527		181	189	198	206	214	222	230	239	247	255	7 5.6
528		263	272	280	288	296	304	313	321	329	337	8 6.4
529		346	354	362	370	378	387	395	403	411	419	9 7.2
530		428	436	444	452	460	469	477	485	493	501	
531		509	518	526	534	542	550	558	567	575	583	
532		591	599	607	616	624	632	640	648	656	665	
533		673	681	689	697	705	713	722	730	738	746	
534		754	762	770	779	787	795	803	811	819	827	
535		835	843	852	860	868	876	884	892	900	908	7
536		916	925	933	941	949	957	965	973	981	989	1 0.7
537		997	*006	*014	*022	*030	*038	*046	*054	*062	*070	2 1.4
538	73	078	086	094	102	111	119	127	135	143	151	3 2.1
539		159	167	175	183	191	199	207	215	223	231	4 2.8
540		239	247	255	263	272	280	288	296	304	312	5 3.5
541		320	328	336	344	352	360	368	376	384	392	6 4.2
542		400	408	416	424	432	440	448	456	464	472	7 4.9
543		480	488	496	504	512	520	528	536	544	552	8 5.6
544		560	568	576	584	592	600	608	616	624	632	9 6.3
545		640	648	656	664	672	679	687	695	703	711	
546		719	727	735	743	751	759	767	775	783	791	
547		799	807	815	823	830	838	846	854	862	870	
548		878	886	894	902	910	918	926	933	941	949	
549		957	965	973	981	989	997	*005	*013	*021	*028	
550	74	036	044	052	060	068	076	084	092	099	107	
N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.
	S.'	T.'					S.''	T.''				S.'' T.
5'	6.46	373	373				0° 8' = 480''	4.68 557	558	1° 26' = 5160''	4.68 553	56
6		373	373				0 9 = 540	557	558	1 27 = 5220	553	56
50		371	376				0 10 = 600	557	558	1 28 = 5280	553	56
55		371	376				1 23 = 4980	553	566	1 29 = 5340	553	56
							1 24 = 5040	553	566	1 30 = 5400	553	56
							1 25 = 5100	553	566	1 31 = 5460	552	56
							1 26 = 5160	553	567	1 32 = 5520	552	56

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.		
550	74	036	044	052	060	068	076	084	092	099	107			
551		115	123	131	139	147	155	162	170	178	186			
552		194	202	210	218	225	233	241	249	257	265			
553		273	280	288	296	304	312	320	327	335	343			
554		351	359	367	374	382	390	398	406	414	421			
555		429	437	445	453	461	468	476	484	492	500			
556		507	515	523	531	539	547	554	562	570	578			
557		586	593	601	609	617	624	632	640	648	656			
558		663	671	679	687	695	702	710	718	726	733			
559		741	749	757	764	772	780	788	796	803	811			
560		819	827	834	842	850	858	865	873	881	889	8		
561		896	904	912	920	927	935	943	950	958	966	1 0.8		
562		974	981	989	997	*005	*012	*020	*028	*035	*043	2 1.6		
563	75	051	059	066	074	082	089	097	105	113	120	3 2.4		
564		128	136	143	151	159	166	174	182	189	197	4 3.2		
565		205	213	220	228	236	243	251	259	266	274	5 4.0		
566		282	289	297	305	312	320	328	335	343	351	6 4.8		
567		358	366	374	381	389	397	404	412	420	427	7 5.6		
568		435	442	450	458	465	473	481	488	496	504	8 6.4		
569		511	519	526	534	542	549	557	565	572	580	9 7.2		
570		587	595	603	610	618	626	633	641	648	656			
571		664	671	679	686	694	702	709	717	724	732			
572		740	747	755	762	770	778	785	793	800	808			
573		815	823	831	838	846	853	861	868	876	884			
574		891	899	906	914	921	929	937	944	952	959			
575		967	974	982	989	997	*005	*012	*020	*027	*035			
576	76	042	050	057	065	072	080	087	095	103	110			
577		118	125	133	140	148	155	163	170	178	185			
578		193	200	208	215	223	230	238	245	253	260			
579		268	275	283	290	298	305	313	320	328	335			
580		343	350	358	365	373	380	388	395	403	410	7		
581		418	425	433	440	448	455	462	470	477	485	1 0.7		
582		492	500	507	515	522	530	537	545	552	559	2 1.4		
583		567	574	582	589	597	604	612	619	626	634	3 2.1		
584		641	649	656	664	671	678	686	693	701	708	4 2.8		
585		716	723	730	738	745	753	760	768	775	782	5 3.5		
586		790	797	805	812	819	827	834	842	849	856	6 4.2		
587		864	871	879	886	893	901	908	916	923	930	7 4.9		
588		938	945	953	960	967	975	982	989	997	*004	8 5.6		
589	77	012	019	026	034	041	048	056	063	070	078	9 6.3		
590		085	093	100	107	115	122	129	137	144	151			
591		159	166	173	181	188	195	203	210	217	225			
592		232	240	247	254	262	269	276	283	291	298			
593		305	313	320	327	335	342	349	357	364	371			
594		379	386	393	401	408	415	422	430	437	444			
595		452	459	466	474	481	488	495	503	510	517			
596		525	532	539	546	554	561	568	576	583	590			
597		597	605	612	619	627	634	641	648	656	663			
598		670	677	685	692	699	706	714	721	728	735			
599		743	750	757	764	772	779	786	793	801	808			
600		815	822	830	837	844	851	859	866	873	880			
N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.		
	S.	T.					S.	T.				S.	T.	
	373	373		0° 9' =	540"	4.68	557	558		1° 35' =	5700"	4.68	552	569
	371	376		0 10 =	600		557	558		1 36 =	5760		552	569
	371	376		1 31 =	5460		552	568		1 37 =	5820		552	569
	371	377		1 32 =	5520		552	568		1 38 =	5880		552	569
	371	377		1 33 =	5580		552	568		1 39 =	5940		551	569
	370	377		1 34 =	5640		552	568		1 40 =	6000		551	570
	370	377		1 35 =	5700		552	569						

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.	
600	77	815	822	830	837	844	851	859	866	873	880		
601		887	895	902	909	916	924	931	938	945	952		
602		960	967	974	981	988	996	*003	*010	*017	*025		
603	78	032	039	046	053	061	068	075	082	089	097		
604		104	111	118	125	132	140	147	154	161	168		
605		176	183	190	197	204	211	219	226	233	240		
606		247	254	262	269	276	283	290	297	305	312	8	
607		319	326	333	340	347	355	362	369	376	383	1 0.8	
608		390	398	405	412	419	426	433	440	447	455	2 1.6	
609		462	469	476	483	490	497	504	512	519	526	3 2.4	
610		533	540	547	554	561	569	576	583	590	597	4 3.2	
611		604	611	618	625	633	640	647	654	661	668	5 4.0	
612		675	682	689	696	704	711	718	725	732	739	6 4.8	
613		746	753	760	767	774	781	789	796	803	810	7 5.6	
614		817	824	831	838	845	852	859	866	873	880	8 6.4	
615		888	895	902	909	916	923	930	937	944	951	9 7.2	
616		958	965	972	979	986	993	*000	*007	*014	*021		
617	79	029	036	043	050	057	064	071	078	085	092		
618		099	106	113	120	127	134	141	148	155	162		
619		169	176	183	190	197	204	211	218	225	232		
620		239	246	253	260	267	274	281	288	295	302	7	
621		309	316	323	330	337	344	351	358	365	372	1 0.7	
622		379	386	393	400	407	414	421	428	435	442	2 1.4	
623		449	456	463	470	477	484	491	498	505	511	3 2.1	
624		518	525	532	539	546	553	560	567	574	581	4 2.8	
625		588	595	602	609	616	623	630	637	644	650	5 3.5	
626		657	664	671	678	685	692	699	706	713	720	6 4.2	
627		727	734	741	748	754	761	768	775	782	789	7 4.9	
628		796	803	810	817	824	831	837	844	851	858	8 5.6	
629		865	872	879	886	893	900	906	913	920	927	9 6.3	
630		934	941	948	955	962	969	975	982	989	996		
631	80	003	010	017	024	030	037	044	051	058	065		
632		072	079	085	092	099	106	113	120	127	134		
633		140	147	154	161	168	175	182	188	195	202		
634		209	216	223	229	236	243	250	257	264	271		
635		277	284	291	298	305	312	318	325	332	339		
636		346	353	359	366	373	380	387	393	400	407	8	
637		414	421	428	434	441	448	455	462	468	475	1 0.6	
638		482	489	496	502	509	516	523	530	536	543	2 1.2	
639		550	557	564	570	577	584	591	598	604	611	3 1.8	
640		618	625	632	638	645	652	659	665	672	679	4 2.4	
641		686	693	699	706	713	720	726	733	740	747	5 3.0	
642		754	760	767	774	781	787	794	801	808	814	6 3.6	
643		821	828	835	841	848	855	862	868	875	882	7 4.2	
644		889	895	902	909	916	922	929	936	943	949	8 4.8	
645		956	963	969	976	983	990	996	*003	*010	*017	9 5.4	
646	81	023	030	037	043	050	057	064	070	077	084		
647		090	097	104	111	117	124	131	137	144	151		
648		158	164	171	178	184	191	198	204	211	218		
649		224	231	238	245	251	258	265	271	278	285		
650		291	298	305	311	318	325	331	338	345	351		
N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.	
		S.	T.				S.	T.				S.	T.
6'	6.46	373	373		0° 10' = 600"	4.68	557	558		1° 44' = 6240"	4.68	551	51
7		373	373		0 11 = 660		557	558		1 45 = 6300		551	51
60		370	377		1 40 = 6000		551	570		1 46 = 6360		551	51
63		370	377		1 41 = 6060		551	570		1 47 = 6420		550	51
64		370	378		1 42 = 6120		551	570		1 48 = 6480		550	51
65		370	378		1 43 = 6180		551	570		1 49 = 6540		550	51
					1 44 = 6240		551	571					

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.		
650	81	291	298	303	311	318	323	331	338	343	351			
651		358	365	371	378	385	391	398	405	411	418			
652		423	431	438	445	451	458	465	471	478	485			
653		491	498	505	511	518	525	531	538	544	551			
654		558	564	571	578	584	591	598	604	611	617			
655		624	631	637	644	651	657	664	671	677	684			
656		690	697	704	710	717	723	730	737	743	750			
657		757	763	770	776	783	790	796	803	809	816			
658		823	829	836	842	849	856	862	869	875	882			
659		889	895	902	908	915	921	928	935	941	948			
660		954	961	968	974	981	987	994	*000	*007	*014	7		
661	82	020	027	033	040	046	053	060	066	073	079	1 0.7		
662		086	092	099	105	112	119	125	132	138	145	2 1.4		
663		151	158	164	171	178	184	191	197	204	210	3 2.1		
664		217	223	230	236	243	249	256	263	269	276	4 2.8		
665		282	289	295	302	308	315	321	328	334	341	5 3.5		
666		347	354	360	367	373	380	387	393	400	406	6 4.2		
667		413	419	426	432	439	445	452	458	465	471	7 4.9		
668		478	484	491	497	504	510	517	523	530	536	8 5.6		
669		543	549	556	562	569	575	582	588	595	601	9 6.3		
670		607	614	620	627	633	640	646	653	659	666			
671		672	679	685	692	698	705	711	718	724	730			
672		737	743	750	756	763	769	776	782	789	795			
673		802	808	814	821	827	834	840	847	853	860			
674		866	872	879	885	892	898	905	911	918	924			
675		930	937	943	950	956	963	969	975	982	988			
676		995	*001	*008	*014	*020	*027	*033	*040	*046	*052			
677	83	059	065	072	078	085	091	097	104	110	117			
678		123	129	136	142	149	155	161	168	174	181			
679		187	193	200	206	213	219	225	232	238	245			
680		251	257	264	270	276	283	289	296	302	308	6		
681		313	321	327	334	340	347	353	359	366	372	1 0.6		
682		378	385	391	398	404	410	417	423	429	436	2 1.2		
683		442	448	455	461	467	474	480	487	493	499	3 1.8		
684		506	512	518	525	531	537	544	550	556	563	4 2.4		
685		569	575	582	588	594	601	607	613	620	626	5 3.0		
686		632	639	645	651	658	664	670	677	683	689	6 3.6		
687		696	702	708	715	721	727	734	740	746	753	7 4.2		
688		759	765	771	778	784	790	797	803	809	816	8 4.8		
689		822	828	835	841	847	853	860	866	872	879	9 5.4		
690		885	891	897	904	910	916	923	929	935	942			
691		948	954	960	967	973	979	985	992	998	*004			
692	84	011	017	023	029	036	042	048	055	061	067			
693		073	080	086	092	098	105	111	117	123	130			
694		136	142	148	155	161	167	173	180	186	192			
695		198	205	211	217	223	230	236	242	248	255			
696		261	267	273	280	286	292	298	305	311	317			
697		323	330	336	342	348	354	361	367	373	379			
698		386	392	398	404	410	417	423	429	435	442			
699		448	454	460	466	473	479	485	491	497	504			
700		510	516	522	528	535	541	547	553	559	566			
N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.		
	S.	T.					S.	T.				S.	T.	
646	373	373		0° 10' =	600"	4.68	557	558		1° 51' =	6660"	4.68	530	573
	373	373		0 11 =	660		557	558		1 52 =	6720		530	573
	370	378		0 12 =	720		557	558		1 53 =	6780		530	573
	370	378		1 48 =	6480		550	572		1 54 =	6840		530	573
				1 49 =	6540		550	572		1 55 =	6900		549	574
				1 50 =	6600		550	572		1 56 =	6960		549	574
				1 51 =	6660		530	573		1 57 =	7020		549	574

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
700	84 510	516	522	528	535	541	547	553	559	566	
701	572	578	584	590	597	603	609	615	621	628	
702	634	640	646	652	658	665	671	677	683	689	
703	696	702	708	714	720	726	733	739	745	751	
704	757	763	770	776	782	788	794	800	807	813	
705	819	825	831	837	844	850	856	862	868	874	
706	880	887	893	899	905	911	917	924	930	936	
707	942	948	954	960	967	973	979	985	991	997	
708	85 003	009	016	022	028	034	040	046	052	058	
709	065	071	077	083	089	095	101	107	114	120	
710	126	132	138	144	150	156	163	169	175	181	
711	187	193	199	205	211	217	224	230	236	242	
712	248	254	260	266	272	278	285	291	297	303	
713	309	315	321	327	333	339	345	352	358	364	
714	370	376	382	388	394	400	406	412	418	425	
715	431	437	443	449	455	461	467	473	479	485	
716	491	497	503	509	516	522	528	534	540	546	
717	552	558	564	570	576	582	588	594	600	606	
718	612	618	625	631	637	643	649	655	661	667	
719	673	679	685	691	697	703	709	715	721	727	
720	733	739	745	751	757	763	769	775	781	788	
721	794	800	806	812	818	824	830	836	842	848	
722	854	860	866	872	878	884	890	896	902	908	
723	914	920	926	932	938	944	950	956	962	968	
724	974	980	986	992	998	*004	*010	*016	*022	*028	
725	86 034	040	046	052	058	064	070	076	082	088	
726	094	100	106	112	118	124	130	136	141	147	
727	153	159	165	171	177	183	189	195	201	207	
728	213	219	225	231	237	243	249	255	261	267	
729	273	279	285	291	297	303	308	314	320	326	
730	332	338	344	350	356	362	368	374	380	386	
731	392	398	404	410	415	421	427	433	439	445	
732	451	457	463	469	475	481	487	493	499	504	
733	510	516	522	528	534	540	546	552	558	564	
734	570	576	581	587	593	599	605	611	617	623	
735	629	635	641	646	652	658	664	670	676	682	
736	688	694	700	705	711	717	723	729	735	741	
737	747	753	759	764	770	776	782	788	794	800	
738	806	812	817	823	829	835	841	847	853	859	
739	864	870	876	882	888	894	900	906	911	917	
740	923	929	935	941	947	953	958	964	970	976	
741	982	988	994	999	*005	*011	*017	*023	*029	*035	
742	87 040	046	052	058	064	070	075	081	087	093	
743	099	105	111	116	122	128	134	140	146	151	
744	157	163	169	175	181	186	192	198	204	210	
745	216	221	227	233	239	245	251	256	262	268	
746	274	280	286	291	297	303	309	315	320	326	
747	332	338	344	349	355	361	367	373	379	384	
748	390	396	402	408	413	419	425	431	437	442	
749	448	454	460	466	471	477	483	489	495	500	
750	506	512	518	523	529	535	541	547	552	558	

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.	
		S.'	T.'				S. II'	T. II'				S. II'	T. II'
7'	6.46	373	373	0° 11' =	660"	4.68	557	558	1° 59' =	7140"	4.68	549	575
8		373	373	0 12 =	720		557	558	2 0 =	7200		549	575
70		370	379	0 13 =	780		557	558	2 1 =	7260		549	575
71		370	379	I 56 =	6960		549	574	2 2 =	7320		548	576
72		369	379	I 57 =	7020		549	574	2 3 =	7380		548	576
74		369	379	I 58 =	7080		549	575	2 4 =	7440		548	576
75		369	380	I 59 =	7140		549	575	2 5 =	7500		548	577

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.
750	87	506	512	518	523	529	535	541	547	552	558	
751		564	570	576	581	587	593	599	604	610	616	
752		622	628	633	639	645	651	656	662	668	674	
753		679	685	691	697	703	708	714	720	726	731	
754		737	743	749	754	760	766	772	777	783	789	
755		795	800	806	812	818	823	829	835	841	846	
756		852	858	864	869	875	881	887	892	898	904	
757		910	915	921	927	933	938	944	950	955	961	
758		967	973	978	984	990	996	*001	*007	*013	*018	
759	88	024	030	036	041	047	053	058	064	070	076	
760		081	087	093	098	104	110	116	121	127	133	8
761		138	144	150	156	161	167	173	178	184	190	1 0.6
762		195	201	207	213	218	224	230	235	241	247	2 1.2
763		252	258	264	270	275	281	287	292	298	304	3 1.8
764		309	315	321	326	332	338	343	349	355	360	4 2.4
765		366	372	377	383	389	395	400	406	412	417	5 3.0
766		423	429	434	440	446	451	457	463	468	474	6 3.6
767		480	485	491	497	502	508	513	519	525	530	7 4.2
768		536	542	547	553	559	564	570	576	581	587	8 4.8
769		593	598	604	610	615	621	627	632	638	643	9 5.4
770		649	655	660	666	672	677	683	689	694	700	
771		705	711	717	722	728	734	739	745	750	756	
772		762	767	773	779	784	790	795	801	807	812	
773		818	824	829	835	840	846	852	857	863	868	
774		874	880	885	891	897	902	908	913	919	925	
775		930	936	941	947	953	958	964	969	975	981	
776		986	992	997	*003	*009	*014	*020	*025	*031	*037	
777	89	042	048	053	059	064	070	076	081	087	092	
778		098	104	109	115	120	126	131	137	143	148	
779		154	159	165	170	176	182	187	193	198	204	
780		209	215	221	226	232	237	243	248	254	260	5
781		265	271	276	282	287	293	298	304	310	315	1 0.5
782		321	326	332	337	343	348	354	360	365	371	2 1.0
783		376	382	387	393	398	404	409	415	421	426	3 1.5
784		432	437	443	448	454	459	465	470	476	481	4 2.0
785		487	492	498	504	509	515	520	526	531	537	5 2.5
786		542	548	553	559	564	570	575	581	586	592	6 3.0
787		597	603	609	614	620	625	631	636	642	647	7 3.5
788		653	658	664	669	675	680	686	691	697	702	8 4.0
789		708	713	719	724	730	735	741	746	752	757	9 4.5
790		763	768	774	779	785	790	796	801	807	812	
791		818	823	829	834	840	845	851	856	862	867	
792		873	878	883	889	894	900	905	911	916	922	
793		927	933	938	944	949	955	960	966	971	977	
794		982	988	993	998	*004	*009	*015	*020	*026	*031	
795	90	037	042	048	053	059	064	069	075	080	086	
796		091	097	102	108	113	119	124	129	135	140	
797		146	151	157	162	168	173	179	184	189	195	
798		200	206	211	217	222	227	233	238	244	249	
799		255	260	266	271	276	282	287	293	298	304	
800		309	314	320	325	331	336	342	347	352	358	
N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.
		S.	T.			S.	T.			S.	T.	
7'	6.46	373	373			0° 12' = 720''	4.68	557	558	2° 8' = 7680''	4.68	547 578
8		373	373			0 13 = 780		557	558	2 9 = 7740		547 578
75	369	380				0 14 = 840		557	558	2 10 = 7800		547 578
80	369	380				2 5 = 7500		548	577	2 11 = 7860		547 579
						2 6 = 7560		548	577	2 12 = 7920		547 579
						2 7 = 7620		548	577	2 13 = 7980		547 579
						2 8 = 7680		547	578	2 14 = 8040		546 579

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.
800	90	309	314	320	325	331	336	342	347	352	358	
801		363	369	374	380	385	390	396	401	407	412	
802		417	423	428	434	439	445	450	455	461	466	
803		472	477	482	488	493	499	504	509	515	520	
804		526	531	536	542	547	553	558	563	569	574	
805		580	585	590	596	601	607	612	617	623	628	
806		634	639	644	650	655	660	666	671	677	682	
807		687	693	698	703	709	714	720	725	730	736	
808		741	747	752	757	763	768	773	779	784	789	
809		795	800	806	811	816	822	827	832	838	843	
810		849	854	859	865	870	875	881	886	891	897	
811		902	907	913	918	924	929	934	940	945	950	
812		956	961	966	972	977	982	988	993	998	*004	1 0.6
813	91	009	014	020	025	030	036	041	046	052	057	2 1.2
814		062	068	073	078	084	089	094	100	105	110	3 1.8
815		116	121	126	132	137	142	148	153	158	164	4 2.4
816		169	174	180	185	190	196	201	206	212	217	5 3.0
817		222	228	233	238	243	249	254	259	265	270	6 3.6
818		275	281	286	291	297	302	307	312	318	323	7 4.2
819		328	334	339	344	350	355	360	365	371	376	8 4.8
820		381	387	392	397	403	408	413	418	424	429	9 5.4
821		434	440	445	450	455	461	466	471	477	482	
822		487	492	498	503	508	514	519	524	529	535	
823		540	545	551	556	561	566	572	577	582	587	
824		593	598	603	609	614	619	624	630	635	640	
825		645	651	656	661	666	672	677	682	687	693	
826		698	703	709	714	719	724	730	735	740	745	
827		751	756	761	766	772	777	782	787	793	798	
828		803	808	814	819	824	829	834	840	845	850	
829		855	861	866	871	876	882	887	892	897	903	
830		908	913	918	924	929	934	939	944	950	955	
831		960	965	971	976	981	986	991	997	*002	*007	
832	92	012	018	023	028	033	038	044	049	054	059	1 0.5
833		065	070	075	080	085	091	096	101	106	111	2 1.0
834		117	122	127	132	137	143	148	153	158	163	3 1.5
835		169	174	179	184	189	195	200	205	210	215	4 2.0
836		221	226	231	236	241	247	252	257	262	267	5 2.5
837		273	278	283	288	293	298	304	309	314	319	6 3.0
838		324	330	335	340	345	350	355	361	366	371	7 3.5
839		376	381	387	392	397	402	407	412	418	423	8 4.0
840		428	433	438	443	449	454	459	464	469	474	9 4.5
841		480	485	490	495	500	505	511	516	521	526	
842		531	536	542	547	552	557	562	567	572	578	
843		583	588	593	598	603	609	614	619	624	629	
844		634	639	645	650	655	660	665	670	675	681	
845		686	691	696	701	706	711	716	722	727	732	
846		737	742	747	752	758	763	768	773	778	783	
847		788	793	799	804	809	814	819	824	829	834	
848		840	845	850	855	860	865	870	875	881	886	
849		891	896	901	906	911	916	921	927	932	937	
850		942	947	952	957	962	967	973	978	983	988	
N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.
		S.' T.'					S.' T.'					S.' T.'
8'	6.46	373	373		0° 13' = 780''	4.68	557	558	2° 16' = 8160''	4.68	546	580
9		373	373		0° 14' = 840		557	558	2° 17' = 8220		546	580
					0° 15' = 900		557	558	2° 18' = 8280		546	581
80		369	380		2° 13' = 7980		547	579	2° 19' = 8340		546	581
81		369	381		2° 14' = 8040		546	579	2° 20' = 8400		545	582
82		368	381		2° 15' = 8100		546	580	2° 21' = 8460		545	582
85		368	381		2° 16' = 8160		546	580	2° 22' = 8520		545	582

N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
850	92 942	947	952	957	962	967	973	978	983	988	
851	993	998	*003	*008	*013	*018	*024	*029	*034	*039	
852	93 044	049	054	059	064	069	075	080	085	090	
853	095	100	105	110	115	120	125	131	136	141	
854	146	151	156	161	166	171	176	181	186	192	
855	197	202	207	212	217	222	227	232	237	242	
856	247	252	258	263	268	273	278	283	288	293	
857	298	303	308	313	318	323	328	334	339	344	
858	349	354	359	364	369	374	379	384	389	394	
859	399	404	409	414	420	425	430	435	440	445	
860	450	455	460	465	470	475	480	485	490	495	
861	500	505	510	515	520	526	531	536	541	546	
862	551	556	561	566	571	576	581	586	591	596	
863	601	606	611	616	621	626	631	636	641	646	
864	651	656	661	666	671	676	682	687	692	697	
865	702	707	712	717	722	727	732	737	742	747	
866	752	757	762	767	772	777	782	787	792	797	
867	802	807	812	817	822	827	832	837	842	847	
868	852	857	862	867	872	877	882	887	892	897	
869	902	907	912	917	922	927	932	937	942	947	
870	952	957	962	967	972	977	982	987	992	997	
871	94 002	007	012	017	022	027	032	037	042	047	
872	052	057	062	067	072	077	082	086	091	096	
873	101	106	111	116	121	126	131	136	141	146	
874	151	156	161	166	171	176	181	186	191	196	
875	201	206	211	216	221	226	231	236	240	245	
876	250	255	260	265	270	275	280	285	290	295	
877	300	305	310	315	320	325	330	335	340	345	
878	349	354	359	364	369	374	379	384	389	394	
879	399	404	409	414	419	424	429	433	438	443	
880	448	453	458	463	468	473	478	483	488	493	
881	498	503	507	512	517	522	527	532	537	542	
882	547	552	557	562	567	571	576	581	586	591	
883	596	601	606	611	616	621	626	630	635	640	
884	645	650	655	660	665	670	675	680	685	689	
885	694	699	704	709	714	719	724	729	734	738	
886	743	748	753	758	763	768	773	778	783	787	
887	792	797	802	807	812	817	822	827	832	836	
888	841	846	851	856	861	866	871	876	880	885	
889	890	895	900	905	910	915	919	924	929	934	
890	939	944	949	954	959	963	968	973	978	983	
891	988	993	998	*002	*007	*012	*017	*022	*027	*032	
892	95 036	041	046	051	056	061	066	071	075	080	
893	085	090	095	100	105	109	114	119	124	129	
894	134	139	143	148	153	158	163	168	173	177	
895	182	187	192	197	202	207	211	216	221	226	
896	231	236	240	245	250	255	260	265	270	274	
897	279	284	289	294	299	303	308	313	318	323	
898	328	332	337	342	347	352	357	361	366	371	
899	376	381	386	390	395	400	405	410	415	419	
900	424	429	434	439	444	448	453	458	463	468	
N.	L. 0	1	2	3	4	5	6	7	8	9	P. P.
	S.' T.'					S.'' T.'					S.'' T.'
8'	6.46 373	373		0° 14' = 840''	4.68 557	558		2° 25' = 8700''	4.68 545	583	
9	373	373		0 15 = 900	557	558		2 26 = 8760	544	584	
85	368	381		2 21 = 8460	545	582		2 27 = 8820	544	584	
86	368	382		2 22 = 8520	545	582		2 28 = 8880	544	584	
89	368	382		2 23 = 8580	545	583		2 29 = 8940	544	585	
90	368	383		2 24 = 8640	545	583		2 30 = 9000	544	585	
				2 25 = 8700	545	583					

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.	
900	95	424	429	434	439	444	448	453	458	463	468		
901		472	477	482	487	492	497	501	506	511	516		
902		521	525	530	535	540	545	550	554	559	564		
903		569	574	578	583	588	593	598	602	607	612		
904		617	622	626	631	636	641	646	650	655	660		
905		665	670	674	679	684	689	694	698	703	708		
906		713	718	722	727	732	737	742	746	751	756		
907		761	766	770	775	780	785	789	794	799	804		
908		809	813	818	823	828	832	837	842	847	852		
909		856	861	866	871	875	880	885	890	895	899		
910		904	909	914	918	923	928	933	938	942	947	5	
911		952	957	961	966	971	976	980	985	990	995	1 0.5	
912		999	*004	*009	*014	*019	*023	*028	*033	*038	*042	2 1.0	
913	96	047	052	057	061	066	071	076	080	085	090	3 1.5	
914		095	099	104	109	114	118	123	128	133	137	4 2.0	
915		142	147	152	156	161	166	171	175	180	185	5 2.5	
916		190	194	199	204	209	213	218	223	227	232	6 3.0	
917		237	242	246	251	256	261	265	270	275	280	7 3.5	
918		284	289	294	298	303	308	313	317	322	327	8 4.0	
919		332	336	341	346	350	355	360	365	369	374	9 4.5	
920		379	384	388	393	398	402	407	412	417	421		
921		426	431	435	440	445	450	454	459	464	468		
922		473	478	483	487	492	497	501	506	511	515		
923		520	525	530	534	539	544	548	553	558	562		
924		567	572	577	581	586	591	595	600	605	609		
925		614	619	624	628	633	638	642	647	652	656		
926		661	666	670	675	680	685	689	694	699	703		
927		708	713	717	722	727	731	736	741	745	750		
928		755	759	764	769	774	778	783	788	792	797		
929		802	806	811	816	820	825	830	834	839	844		
930		848	853	858	862	867	872	876	881	886	890	4	
931		895	900	904	909	914	918	923	928	932	937	1 0.4	
932		942	946	951	956	960	965	970	974	979	984	2 0.8	
933		988	993	997	*002	*007	*011	*016	*021	*025	*030	3 1.2	
934	97	035	039	044	049	053	058	063	067	072	077	4 1.6	
935		081	086	090	095	100	104	109	114	118	123	5 2.0	
936		128	132	137	142	146	151	155	160	165	169	6 2.4	
937		174	179	183	188	192	197	202	206	211	216	7 2.8	
938		220	225	230	234	239	243	248	253	257	262	8 3.2	
939		267	271	276	280	285	290	294	299	304	308	9 3.6	
940		313	317	322	327	331	336	340	345	350	354		
941		359	364	368	373	377	382	387	391	396	400		
942		405	410	414	419	424	428	433	437	442	447		
943		451	456	460	465	470	474	479	483	488	493		
944		497	502	506	511	516	520	525	529	534	539		
945		543	548	552	557	562	566	571	575	580	585		
946		589	594	598	603	607	612	617	621	626	630		
947		635	640	644	649	653	658	663	667	672	676		
948		681	685	690	695	699	704	708	713	717	722		
949		727	731	736	740	745	749	754	759	763	768		
950		772	777	782	786	791	795	800	804	809	813		
N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.	
		S.	T.				S.	T.				S.	T.
9'	6.46	373	373		0° 15' = 900"	4.68	557	558		2° 34' = 9240"	4.68	543	587
10		373	373		0 16 = 960		557	558		2 35 = 9300		543	587
90		368	383		2 30 = 9000		544	585		2 36 = 9360		543	587
91		368	383		2 31 = 9060		544	585		2 37 = 9420		542	588
92		367	383		2 32 = 9120		543	586		2 38 = 9480		542	588
94		367	383		2 33 = 9180		543	586		2 39 = 9540		542	588
95		367	384		2 34 = 9240		543	587					

N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.	
950	97	772	777	782	786	791	795	800	804	809	813		
951		818	823	827	832	836	841	845	850	855	859		
952		864	868	873	877	882	886	891	896	900	905		
953		909	914	918	923	928	932	937	941	946	950		
954		955	959	964	968	973	978	982	987	991	996		
955	98	000	005	009	014	019	023	028	032	037	041		
956		046	050	055	059	064	068	073	078	082	087		
957		091	096	100	105	109	114	118	123	127	132		
958		137	141	146	150	155	159	164	168	173	177		
959		182	186	191	195	200	204	209	214	218	223		
960		227	232	236	241	245	250	254	259	263	268		
961		272	277	281	286	290	295	299	304	308	313	5	
962		318	322	327	331	336	340	345	349	354	358	1 0.5	
963		363	367	372	376	381	385	390	394	399	403	2 1.0	
964		408	412	417	421	426	430	435	439	444	448	3 1.5	
965		453	457	462	466	471	475	480	484	489	493	4 2.0	
966		498	502	507	511	516	520	525	529	534	538	5 2.5	
967		543	547	552	556	561	565	570	574	579	583	6 3.0	
968		588	592	597	601	605	610	614	619	623	628	7 3.5	
969		632	637	641	646	650	655	659	664	668	673	8 4.0	
970		677	682	686	691	695	700	704	709	713	717	9 4.5	
971		722	726	731	735	740	744	749	753	758	762		
972		767	771	776	780	784	789	793	798	802	807		
973		811	816	820	825	829	834	838	843	847	851		
974		856	860	865	869	874	878	883	887	892	896		
975		900	905	909	914	918	923	927	932	936	941		
976		945	949	954	958	963	967	972	976	981	985		
977		989	994	998	*003	*007	*012	*016	*021	*025	*029		
978	99	034	038	043	047	052	056	061	065	069	074		
979		078	083	087	092	096	100	105	109	114	118		
980		123	127	131	136	140	145	149	154	158	162	4	
981		167	171	176	180	185	189	193	198	202	207	1 0.4	
982		211	216	220	224	229	233	238	242	247	251	2 0.8	
983		255	260	264	269	273	277	282	286	291	295	3 1.2	
984		300	304	308	313	317	322	326	330	335	339	4 1.6	
985		344	348	352	357	361	366	370	374	379	383	5 2.0	
986		388	392	396	401	405	410	414	419	423	427	6 2.4	
987		432	436	441	445	449	454	458	463	467	471	7 2.8	
988		476	480	484	489	493	498	502	506	511	515	8 3.2	
989		520	524	528	533	537	542	546	550	555	559	9 3.6	
990		564	568	572	577	581	585	590	594	599	603		
991		607	612	616	621	625	629	634	638	642	647		
992		651	656	660	664	669	673	677	682	686	691		
993		695	699	704	708	712	717	721	726	730	734		
994		739	743	747	752	756	760	765	769	774	778		
995		782	787	791	795	800	804	808	813	817	822		
996		826	830	835	839	843	848	852	856	861	865		
997		870	874	878	883	887	891	896	900	904	909		
998		913	917	922	926	930	935	939	944	948	952		
999		957	961	965	970	974	978	983	987	991	996		
1000	00	000	004	009	013	017	022	026	030	035	039		
N.	L.	0	1	2	3	4	5	6	7	8	9	P. P.	
		S.'	T.'				S."	T."				S."	T."
9'	6.46	373	373		0° 15' = 900"	4.68	557	558		2° 41' = 9660"	4.68	542	589
10		373	373		0 16 = 960		557	558		2 42 = 9720		541	590
95		367	384		0 17 = 1020		557	558		2 43 = 9780		541	590
98		367	384		2 38 = 9480		542	588		2 44 = 9840		541	590
99		367	385		2 39 = 9540		542	588		2 45 = 9900		541	591
100		366	385		2 40 = 9600		542	589		2 46 = 9960		541	591
					2 41 = 9660		542	589		2 47 = 10020		540	592

N.	L.	0	1	2	3	4	5	6	7	8	9
1000	000	0000	0434	0869	1303	1737	2171	2605	3039	3473	3907
1001		4341	4775	5208	5642	6076	6510	6943	7377	7810	8244
1002		8677	9111	9544	9977	*0411	*0844	*1277	*1710	*2143	*2576
1003	001	3009	3442	3875	4308	4741	5174	5607	6039	6472	6905
1004		7337	7770	8202	8635	9067	9499	9932	*0364	*0796	*1228
1005	002	1661	2093	2525	2957	3389	3821	4253	4685	5116	5548
1006		5980	6411	6843	7275	7706	8138	8569	9001	9432	9863
1007	003	0295	0726	1157	1588	2019	2451	2882	3313	3744	4174
1008		4605	5036	5467	5898	6328	6759	7190	7620	8051	8481
1009		8912	9342	9772	*0203	*0633	*1063	*1493	*1924	*2354	*2784
1010	004	3214	3644	4074	4504	4933	5363	5793	6223	6652	7082
1011		7512	7941	8371	8800	9229	9659	*0088	*0517	*0947	*1376
1012	005	1805	2234	2663	3092	3521	3950	4379	4808	5237	5666
1013		6094	6523	6952	7380	7809	8238	8666	9094	9523	9951
1014	006	0380	0808	1236	1664	2092	2521	2949	3377	3805	4233
1015		4660	5088	5516	5944	6372	6799	7227	7655	8082	8510
1016		8937	9365	9792	*0219	*0647	*1074	*1501	*1928	*2355	*2782
1017	007	3210	3637	4064	4490	4917	5344	5771	6198	6624	7051
1018		7478	7904	8331	8757	9184	9610	*0037	*0463	*0889	*1316
1019	008	1742	2168	2594	3020	3446	3872	4298	4724	5150	5576
1020		6002	6427	6853	7279	7704	8130	8556	8981	9407	9832
1021	009	0257	0683	1108	1533	1959	2384	2809	3234	3659	4084
1022		4509	4934	5359	5784	6208	6633	7058	7483	7907	8332
1023		8756	9181	9605	*0030	*0454	*0878	*1303	*1727	*2151	*2575
1024	010	3000	3424	3848	4272	4696	5120	5544	5967	6391	6815
1025		7239	7662	8086	8510	8933	9357	9780	*0204	*0627	*1050
1026	011	1474	1897	2320	2743	3166	3590	4013	4436	4859	5282
1027		5704	6127	6550	6973	7396	7818	8241	8664	9086	9509
1028		9931	*0354	*0776	*1198	*1621	*2043	*2465	*2887	*3310	*3732
1029	012	4154	4576	4998	5420	5842	6264	6685	7107	7529	7951
1030		8372	8794	9215	9637	*0059	*0480	*0901	*1323	*1744	*2165
1031	013	2587	3008	3429	3850	4271	4692	5113	5534	5955	6376
1032		6797	7218	7639	8059	8480	8901	9321	9742	*0162	*0583
1033	014	1003	1424	1844	2264	2685	3105	3525	3945	4365	4785
1034		5205	5625	6045	6465	6885	7305	7725	8144	8564	8984
1035		9403	9823	*0243	*0662	*1082	*1501	*1920	*2340	*2759	*3178
1036	015	3598	4017	4436	4855	5274	5693	6112	6531	6950	7369
1037		7788	8206	8625	9044	9462	9881	*0300	*0718	*1137	*1555
1038	016	1974	2392	2810	3229	3647	4065	4483	4901	5319	5737
1039		6155	6573	6991	7409	7827	8245	8663	9080	9498	9916
1040	017	0333	0751	1168	1586	2003	2421	2838	3256	3673	4090
1041		4507	4924	5342	5759	6176	6593	7010	7427	7844	8260
1042		8677	9094	9511	9927	*0344	*0761	*1177	*1594	*2010	*2427
1043	018	2843	3259	3676	4092	4508	4925	5341	5757	6173	6589
1044		7005	7421	7837	8253	8669	9084	9500	9916	*0332	*0747
1045	019	1163	1578	1994	2410	2825	3240	3656	4071	4486	4902
1046		5317	5732	6147	6562	6977	7392	7807	8222	8637	9052
1047		9467	9882	*0296	*0711	*1126	*1540	*1955	*2369	*2784	*3198
1048	020	3613	4027	4442	4856	5270	5684	6099	6513	6927	7341
1049		7755	8169	8583	8997	9411	9824	*0238	*0652	*1066	*1479
1050	021	1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
N.	L.	0	1	2	3	4	5	6	7	8	9
						S."	T."				
2° 46' =	9960'	4.68	541	591	2° 51' = 10260"		4.68	540	593		
2 47 =	10020		540	592	2 52 = 10320			539	594		
2 48 =	10080		540	592	2 53 = 10380			539	594		
2 49 =	10140		540	592	2 54 = 10440			539	595		
2 50 =	10200		540	593	2 55 = 10500			539	595		

N.	L.	0	1	2	3	4	5	6	7	8	9
1050	021	1893	2307	2720	3134	3547	3961	4374	4787	5201	5614
1051		6027	6440	6854	7267	7680	8093	8506	8919	9332	9745
1052	022	0157	0570	0983	1396	1808	2221	2634	3046	3459	3871
1053		4284	4696	5109	5521	5933	6345	6758	7170	7582	7994
1054		8406	8818	9230	9642	*0054	*0466	*0878	*1289	*1701	*2113
1055	023	2525	2936	3348	3759	4171	4582	4994	5405	5817	6228
1056		6639	7050	7462	7873	8284	8695	9106	9517	9928	*0339
1057	024	0750	1161	1572	1982	2393	2804	3214	3625	4036	4446
1058		4857	5267	5678	6088	6498	6909	7319	7729	8139	8549
1059		8960	9370	9780	*0190	*0600	*1010	*1419	*1829	*2239	*2649
1060	025	3059	3468	3878	4288	4697	5107	5516	5926	6335	6744
1061		7154	7563	7972	8382	8791	9200	9609	*0018	*0427	*0836
1062	026	1245	1654	2063	2472	2881	3289	3698	4107	4515	4924
1063		5333	5741	6150	6558	6967	7375	7783	8192	8600	9008
1064		9416	9824	*0233	*0641	*1049	*1457	*1865	*2273	*2680	*3088
1065	027	3496	3904	4312	4719	5127	5535	5942	6350	6757	7165
1066		7572	7979	8387	8794	9201	9609	*0016	*0423	*0830	*1237
1067	028	1644	2051	2458	2865	3272	3679	4086	4492	4899	5306
1068		5713	6119	6526	6932	7339	7745	8152	8558	8964	9371
1069		9777	*0183	*0590	*0996	*1402	*1808	*2214	*2620	*3026	*3432
1070	029	3838	4244	4649	5055	5461	5867	6272	6678	7084	7489
1071		7895	8300	8706	9111	9516	9922	*0327	*0732	*1138	*1543
1072	030	1948	2353	2758	3163	3568	3973	4378	4783	5188	5592
1073		5997	6402	6807	7211	7616	8020	8425	8830	9234	9638
1074	031	0043	0447	0851	1256	1660	2064	2468	2872	3277	3681
1075		4085	4489	4893	5296	5700	6104	6508	6912	7315	7719
1076		8123	8526	8930	9333	9737	*0140	*0544	*0947	*1350	*1754
1077	032	2157	2560	2963	3367	3770	4173	4576	4979	5382	5785
1078		6188	6590	6993	7396	7799	8201	8604	9007	9409	9812
1079	033	0214	0617	1019	1422	1824	2226	2629	3031	3433	3835
1080		4238	4640	5042	5444	5846	6248	6650	7052	7453	7855
1081		8257	8659	9060	9462	9864	*0265	*0667	*1068	*1470	*1871
1082	034	2273	2674	3075	3477	3878	4279	4680	5081	5482	5884
1083		6285	6686	7087	7487	7888	8289	8690	9091	9491	9892
1084	035	0293	0693	1094	1495	1895	2296	2696	3096	3497	3897
1085		4297	4698	5098	5498	5898	6298	6698	7098	7498	7898
1086		8298	8698	9098	9498	9898	*0297	*0697	*1097	*1496	*1896
1087	036	2295	2695	3094	3494	3893	4293	4692	5091	5491	5890
1088		6289	6688	7087	7486	7885	8284	8683	9082	9481	9880
1089	037	0279	0678	1076	1475	1874	2272	2671	3070	3468	3867
1090		4265	4663	5062	5460	5858	6257	6655	7053	7451	7849
1091		8248	8646	9044	9442	9839	*0237	*0635	*1033	*1431	*1829
1092	038	2226	2624	3022	3419	3817	4214	4612	5009	5407	5804
1093		6202	6599	6996	7393	7791	8188	8585	8982	9379	9776
1094	039	0173	0570	0967	1364	1761	2158	2554	2951	3348	3745
1095		4141	4538	4934	5331	5727	6124	6520	6917	7313	7709
1096		8106	8502	8898	9294	9690	*0086	*0482	*0878	*1274	*1670
1097	040	2066	2462	2858	3254	3650	4045	4441	4837	5232	5628
1098		6023	6419	6814	7210	7605	8001	8396	8791	9187	9582
1099		9977	*0372	*0767	*1162	*1557	*1952	*2347	*2742	*3137	*3532
1100	041	3927	4322	4716	5111	5506	5900	6295	6690	7084	7479
N.	L.	0	1	2	3	4	5	6	7	8	9
				S."	T."				S."	T."	
2° 55'		10500"	4.68	539	595		3° 0' = 10800"	4.68	538	597	
2 56 = 10560				539	595		3 1 = 10860		537	598	
2 57 = 10620				538	596		3 2 = 10920		537	598	
2 58 = 10680				538	596		3 3 = 10980		537	599	
2 59 = 10740				538	597		3 4 = 11040		537	599	

	M.	S'. T'.	Sec.	S". T".
		6.46		4.68
0	180	353 412	10800	538 597
1	181	353 413	10860	537 598
2	182	352 413	10920	537 598
3	183	352 414	10980	537 599
4	184	352 414	11040	537 599
5	185	352 415	11100	537 599
6	186	351 415	11160	536 600
7	187	351 415	11220	536 600
8	188	351 416	11280	536 601
9	189	351 416	11340	536 601
10	190	350 417	11400	535 602
11	191	350 417	11460	535 602
12	192	350 418	11520	535 603
13	193	350 418	11580	535 603
14	194	350 419	11640	534 604
15	195	349 419	11700	534 604
16	196	349 420	11760	534 605
17	197	349 420	11820	534 605
18	198	349 421	11880	533 606
19	199	348 421	11940	533 606
20	200	348 422	12000	533 607
21	201	348 422	12060	533 607
22	202	348 423	12120	532 608
23	203	347 423	12180	532 608
24	204	347 424	12240	532 609
25	205	347 424	12300	532 609
26	206	347 425	12360	531 610
27	207	346 425	12420	531 610
28	208	346 426	12480	531 611
29	209	346 426	12540	531 611
30	210	346 427	12600	530 612
31	211	345 427	12660	530 612
32	212	345 428	12720	530 613
33	213	345 428	12780	530 613
34	214	345 429	12840	529 614
35	215	344 429	12900	529 614
36	216	344 430	12960	529 615
37	217	344 430	13020	529 615
38	218	344 431	13080	528 616
39	219	343 431	13140	528 616
40	220	343 432	13200	528 617
41	221	343 432	13260	528 617
42	222	342 433	13320	527 618
43	223	342 434	13380	527 618
44	224	342 434	13440	527 619
45	225	342 435	13500	526 620
46	226	341 435	13560	526 620
47	227	341 436	13620	526 621
48	228	341 436	13680	526 621
49	229	340 437	13740	525 622
50	230	340 437	13800	525 622
51	231	340 438	13860	525 623
52	232	340 439	13920	525 623
53	233	339 439	13980	524 624
54	234	339 440	14040	524 625
55	235	339 440	14100	524 625
56	236	338 441	14160	523 626
57	237	338 441	14220	523 626
58	238	338 442	14280	523 627
59	239	338 443	14340	522 628
60	240	337 443	14400	522 628

	M.	S'. T'.	Sec.	S". T".
		6.46		4.68
0	240	337 443	14400	522 628
1	241	337 444	14460	522 629
2	242	337 444	14520	522 629
3	243	336 445	14580	521 630
4	244	336 446	14640	521 631
5	245	336 446	14700	521 631
6	246	336 447	14760	520 632
7	247	335 447	14820	520 632
8	248	335 448	14880	520 633
9	249	335 449	14940	520 634
10	250	334 449	15000	519 634
11	251	334 450	15060	519 635
12	252	334 450	15120	519 635
13	253	333 451	15180	518 636
14	254	333 452	15240	518 637
15	255	333 452	15300	518 637
16	256	332 453	15360	517 638
17	257	332 454	15420	517 638
18	258	332 454	15480	517 639
19	259	332 455	15540	516 640
20	260	331 456	15600	516 640
21	261	331 456	15660	516 641
22	262	331 457	15720	515 642
23	263	330 457	15780	515 642
24	264	330 458	15840	515 643
25	265	330 459	15900	514 644
26	266	329 459	15960	514 644
27	267	329 460	16020	514 645
28	268	329 461	16080	513 646
29	269	328 461	16140	513 646
30	270	328 462	16200	513 647
31	271	328 463	16260	512 648
32	272	327 463	16320	512 648
33	273	327 464	16380	512 649
34	274	327 465	16440	511 650
35	275	326 465	16500	511 650
36	276	326 466	16560	511 651
37	277	326 467	16620	510 652
38	278	325 467	16680	510 652
39	279	325 468	16740	510 653
40	280	325 469	16800	509 654
41	281	324 469	16860	509 654
42	282	324 470	16920	509 655
43	283	324 471	16980	508 656
44	284	323 472	17040	508 656
45	285	323 473	17100	508 657
46	286	323 473	17160	507 658
47	287	322 474	17220	507 659
48	288	322 474	17280	507 659
49	289	321 475	17340	506 660
50	290	321 476	17400	506 661
51	291	321 477	17460	506 661
52	292	320 477	17520	505 662
53	293	320 478	17580	505 663
54	294	320 479	17640	505 664
55	295	319 479	17700	504 664
56	296	319 480	17760	504 665
57	297	319 481	17820	503 666
58	298	318 482	17880	503 666
59	299	318 482	17940	503 667
60	300	317 483	18000	502 668

II.

THE LOGARITHMS

OF THE

TRIGONOMETRIC FUNCTIONS

FOR EACH MINUTE.

Formulas for the Use of the Auxiliaries *S* and *T*.

1. When α is in the first five degrees of the quadrant:

$\log \sin \alpha = \log a' + S.'$ $\log \tan \alpha = \log a' + T.'$ $\log \cot \alpha = \text{cpl log tan } \alpha.$		$\log a' = \log \sin \alpha + \text{cpl } S.'$ $= \log \tan \alpha + \text{cpl } T.'$ $= \text{cpl log cot } \alpha + \text{cpl } T.'$
$\log \sin \alpha = \log a'' + S''$ $\log \tan \alpha = \log a'' + T''$ $\log \cot \alpha = \text{cpl log tan } \alpha.$		$\log a'' = \log \sin \alpha + \text{cpl } S''$ $= \log \tan \alpha + \text{cpl } T''$ $= \text{cpl log cot } \alpha + \text{cpl } T''$

2. When α is in the last five degrees of the quadrant:

$\log \cos \alpha = \log(90^\circ - \alpha)' + S.'$ $\log \cot \alpha = \log(90^\circ - \alpha)' + T.'$ $\log \tan \alpha = \text{cpl log cot } \alpha.$		$\log(90^\circ - \alpha)' = \log \cos \alpha + \text{cpl } S.'$ $= \log \cot \alpha + \text{cpl } T.'$ $= \text{cpl log tan } \alpha + \text{cpl } T.'$
$\log \cos \alpha = \log(90^\circ - \alpha)'' + S''$ $\log \cot \alpha = \log(90^\circ - \alpha)'' + T''$ $\log \tan \alpha = \text{cpl log cot } \alpha.$		$\log(90^\circ - \alpha)'' = \log \cos \alpha + \text{cpl } S''$ $= \log \cot \alpha + \text{cpl } T''$ $= \text{cpl log tan } \alpha + \text{cpl } T''$

$$\alpha = 90^\circ - (90^\circ - \alpha).$$

"	'	L. Sin.	d.	Cpl. S'.	Cpl. T'.	L. Tan.	c. d.	L. Cot.	L. Cos.	
0	0	—	—	—	—	—	—	—	0.00 000	60
60	1	6.46 373	30103	3.53 627	3.53 627	6.46 373	30103	3.53 627	0.00 000	59
120	2	6.76 476	17609	3.53 627	3.53 627	6.76 476	17609	3.23 524	0.00 000	58
180	3	6.94 085	12494	3.53 627	3.53 627	6.94 085	12494	3.05 915	0.00 000	57
240	4	7.06 579	9691	3.53 627	3.53 627	7.06 579	9691	2.93 421	0.00 000	56
300	5	7.16 270	7918	3.53 627	3.53 627	7.16 270	7918	2.83 730	0.00 000	55
360	6	7.24 188	6694	3.53 627	3.53 627	7.24 188	6694	2.75 812	0.00 000	54
420	7	7.30 882	5800	3.53 627	3.53 627	7.30 882	5800	2.69 118	0.00 000	53
480	8	7.36 682	5115	3.53 627	3.53 627	7.36 682	5115	2.63 318	0.00 000	52
540	9	7.41 797	4576	3.53 627	3.53 627	7.41 797	4576	2.58 203	0.00 000	51
600	10	7.46 373	4139	3.53 627	3.53 627	7.46 373	4139	2.53 627	0.00 000	50
660	11	7.50 512	3779	3.53 627	3.53 627	7.50 512	3779	2.49 488	0.00 000	49
720	12	7.54 291	3476	3.53 627	3.53 627	7.54 291	3476	2.45 709	0.00 000	48
780	13	7.57 767	3219	3.53 627	3.53 627	7.57 767	3219	2.42 233	0.00 000	47
840	14	7.60 985	2997	3.53 628	3.53 627	7.60 986	2996	2.39 014	0.00 000	46
900	15	7.63 982	2802	3.53 628	3.53 627	7.63 982	2803	2.36 018	0.00 000	45
960	16	7.66 784	2633	3.53 628	3.53 627	7.66 785	2633	2.33 215	0.00 000	44
1020	17	7.69 417	2483	3.53 628	3.53 627	7.69 418	2482	2.30 582	9.99 999	43
1080	18	7.71 900	2348	3.53 628	3.53 627	7.71 900	2348	2.28 100	9.99 999	42
1140	19	7.74 248	2227	3.53 628	3.53 627	7.74 248	2228	2.25 752	9.99 999	41
1200	20	7.76 475	2119	3.53 628	3.53 627	7.76 476	2119	2.23 524	9.99 999	40
1260	21	7.78 594	2021	3.53 628	3.53 627	7.78 595	2020	2.21 405	9.99 999	39
1320	22	7.80 613	1930	3.53 628	3.53 627	7.80 615	1931	2.19 385	9.99 999	38
1380	23	7.82 545	1848	3.53 628	3.53 627	7.82 546	1848	2.17 454	9.99 999	37
1440	24	7.84 393	1773	3.53 628	3.53 627	7.84 394	1773	2.15 606	9.99 999	36
1500	25	7.86 166	1704	3.53 628	3.53 627	7.86 167	1704	2.13 833	9.99 999	35
1560	26	7.87 870	1639	3.53 628	3.53 627	7.87 871	1639	2.12 129	9.99 999	34
1620	27	7.89 509	1579	3.53 628	3.53 626	7.89 510	1579	2.10 490	9.99 999	33
1680	28	7.91 088	1524	3.53 628	3.53 626	7.91 089	1524	2.08 911	9.99 999	32
1740	29	7.92 612	1472	3.53 628	3.53 626	7.92 613	1473	2.07 387	9.99 998	31
1800	30	7.94 084	1424	3.53 628	3.53 626	7.94 086	1424	2.05 914	9.99 998	30
1860	31	7.95 508	1379	3.53 628	3.53 626	7.95 510	1379	2.04 490	9.99 998	29
1920	32	7.96 887	1336	3.53 628	3.53 626	7.96 889	1336	2.03 111	9.99 998	28
1980	33	7.98 223	1297	3.53 628	3.53 626	7.98 225	1297	2.01 775	9.99 998	27
2040	34	7.99 520	1259	3.53 628	3.53 626	7.99 522	1259	2.00 478	9.99 998	26
2100	35	8.00 779	1223	3.53 628	3.53 626	8.00 781	1223	1.99 219	9.99 998	25
2160	36	8.02 002	1190	3.53 628	3.53 626	8.02 004	1190	1.97 996	9.99 998	24
2220	37	8.03 192	1158	3.53 628	3.53 626	8.03 194	1159	1.96 806	9.99 997	23
2280	38	8.04 350	1128	3.53 628	3.53 626	8.04 353	1128	1.95 647	9.99 997	22
2340	39	8.05 478	1100	3.53 628	3.53 626	8.05 481	1100	1.94 519	9.99 997	21
2400	40	8.06 578	1072	3.53 628	3.53 625	8.06 581	1072	1.93 419	9.99 997	20
2460	41	8.07 650	1046	3.53 628	3.53 625	8.07 653	1047	1.92 347	9.99 997	19
2520	42	8.08 696	1022	3.53 628	3.53 625	8.08 700	1022	1.91 300	9.99 997	18
2580	43	8.09 718	999	3.53 629	3.53 625	8.09 722	998	1.90 278	9.99 997	17
2640	44	8.10 717	976	3.53 629	3.53 625	8.10 720	976	1.89 280	9.99 996	16
2700	45	8.11 693	954	3.53 629	3.53 625	8.11 696	955	1.88 304	9.99 996	15
2760	46	8.12 647	934	3.53 629	3.53 625	8.12 651	934	1.87 349	9.99 996	14
2820	47	8.13 581	914	3.53 629	3.53 625	8.13 585	915	1.86 413	9.99 996	13
2880	48	8.14 495	896	3.53 629	3.53 625	8.14 500	895	1.85 500	9.99 996	12
2940	49	8.15 391	877	3.53 629	3.53 624	8.15 395	878	1.84 605	9.99 996	11
3000	50	8.16 268	860	3.53 629	3.53 624	8.16 273	860	1.83 727	9.99 995	10
3060	51	8.17 128	843	3.53 629	3.53 624	8.17 133	843	1.82 867	9.99 995	9
3120	52	8.17 971	827	3.53 629	3.53 624	8.17 976	828	1.82 024	9.99 995	8
3180	53	8.18 798	812	3.53 629	3.53 624	8.18 804	812	1.81 196	9.99 994	7
3240	54	8.19 610	797	3.53 629	3.53 624	8.19 616	797	1.80 384	9.99 995	6
3300	55	8.20 407	782	3.53 629	3.53 624	8.20 413	782	1.79 587	9.99 994	5
3360	56	8.21 189	769	3.53 629	3.53 624	8.21 195	769	1.78 805	9.99 994	4
3420	57	8.21 958	755	3.53 629	3.53 623	8.21 964	756	1.78 036	9.99 994	3
3480	58	8.22 713	743	3.53 629	3.53 623	8.22 720	742	1.77 280	9.99 994	2
3540	59	8.23 456	730	3.53 630	3.53 623	8.23 462	730	1.76 538	9.99 994	1
3600	60	8.24 186		3.53 630	3.53 623	8.24 192		1.75 808	9.99 993	
		L. Cos.	d.			L. Cot.	c. d.	L. Tan.	L. Sin.	

"	'	L. Sin.	d.	Cpl. S'.	Cpl. T'.	L. Tan.	c. d.	L. Cot.	L. Cos.	
3600	0	8.24 186		3-53 630	3-53 623	8.24 192		1.75 808	9.99 993	60
3660	1	8.24 903	717	3-53 630	3-53 623	8.24 910	718	1.75 090	9.99 993	59
3720	2	8.25 609	706	3-53 630	3-53 623	8.25 616	706	1.74 384	9.99 993	58
3780	3	8.26 304	695	3-53 630	3-53 623	8.26 312	696	1.73 688	9.99 993	57
3840	4	8.26 988	684	3-53 630	3-53 622	8.26 996	684	1.73 004	9.99 992	56
3900	5	8.27 661	673	3-53 630	3-53 622	8.27 669	673	1.72 331	9.99 992	55
3960	6	8.28 324	663	3-53 630	3-53 622	8.28 332	663	1.71 668	9.99 992	54
4020	7	8.28 977	653	3-53 630	3-53 622	8.28 986	654	1.71 014	9.99 992	53
4080	8	8.29 621	644	3-53 630	3-53 622	8.29 629	643	1.70 371	9.99 992	52
4140	9	8.30 255	634	3-53 630	3-53 622	8.30 263	634	1.69 737	9.99 991	51
4200	10	8.30 879	624	3-53 630	3-53 621	8.30 888	625	1.69 112	9.99 991	50
4260	11	8.31 495	616	3-53 630	3-53 621	8.31 505	617	1.68 495	9.99 991	49
4320	12	8.32 103	608	3-53 631	3-53 621	8.32 112	607	1.67 888	9.99 990	48
4380	13	8.32 702	599	3-53 631	3-53 621	8.32 711	599	1.67 289	9.99 990	47
4440	14	8.33 292	590	3-53 631	3-53 621	8.33 302	591	1.66 698	9.99 990	46
4500	15	8.33 875	583	3-53 631	3-53 620	8.33 886	584	1.66 114	9.99 990	45
4560	16	8.34 450	575	3-53 631	3-53 620	8.34 461	575	1.65 539	9.99 989	44
4620	17	8.35 018	568	3-53 631	3-53 620	8.35 029	568	1.64 971	9.99 989	43
4680	18	8.35 578	560	3-53 631	3-53 620	8.35 590	561	1.64 410	9.99 989	42
4740	19	8.36 131	553	3-53 631	3-53 620	8.36 143	553	1.63 857	9.99 989	41
4800	20	8.36 678	547	3-53 631	3-53 620	8.36 689	546	1.63 311	9.99 988	40
4860	21	8.37 217	539	3-53 631	3-53 619	8.37 229	540	1.62 771	9.99 988	39
4920	22	8.37 750	533	3-53 632	3-53 619	8.37 762	533	1.62 238	9.99 988	38
4980	23	8.38 276	526	3-53 632	3-53 619	8.38 289	527	1.61 711	9.99 987	37
5040	24	8.38 796	520	3-53 632	3-53 619	8.38 809	520	1.61 191	9.99 987	36
5100	25	8.39 310	514	3-53 632	3-53 619	8.39 323	514	1.60 677	9.99 987	35
5160	26	8.39 818	508	3-53 632	3-53 618	8.39 832	509	1.60 168	9.99 986	34
5220	27	8.40 320	502	3-53 632	3-53 618	8.40 334	502	1.59 666	9.99 986	33
5280	28	8.40 816	496	3-53 632	3-53 618	8.40 830	496	1.59 170	9.99 986	32
5340	29	8.41 307	491	3-53 632	3-53 618	8.41 321	491	1.58 679	9.99 985	31
5400	30	8.41 792	485	3-53 632	3-53 617	8.41 807	486	1.58 193	9.99 985	30
5460	31	8.42 272	480	3-53 632	3-53 617	8.42 287	480	1.57 713	9.99 985	29
5520	32	8.42 746	474	3-53 633	3-53 617	8.42 762	475	1.57 238	9.99 984	28
5580	33	8.43 216	470	3-53 633	3-53 617	8.43 232	470	1.56 768	9.99 984	27
5640	34	8.43 680	464	3-53 633	3-53 617	8.43 696	464	1.56 304	9.99 984	26
5700	35	8.44 139	459	3-53 633	3-53 616	8.44 156	460	1.55 844	9.99 983	25
5760	36	8.44 594	455	3-53 633	3-53 616	8.44 611	455	1.55 389	9.99 983	24
5820	37	8.45 044	450	3-53 633	3-53 616	8.45 061	450	1.54 939	9.99 983	23
5880	38	8.45 489	445	3-53 633	3-53 616	8.45 507	446	1.54 493	9.99 982	22
5940	39	8.45 930	441	3-53 633	3-53 615	8.45 948	441	1.54 052	9.99 982	21
6000	40	8.46 366	436	3-53 634	3-53 615	8.46 385	437	1.53 615	9.99 982	20
6060	41	8.46 799	433	3-53 634	3-53 615	8.46 817	432	1.53 183	9.99 981	19
6120	42	8.47 226	427	3-53 634	3-53 615	8.47 245	428	1.52 755	9.99 981	18
6180	43	8.47 650	424	3-53 634	3-53 614	8.47 669	424	1.52 331	9.99 981	17
6240	44	8.48 069	419	3-53 634	3-53 614	8.48 089	420	1.51 911	9.99 980	16
6300	45	8.48 485	416	3-53 634	3-53 614	8.48 505	416	1.51 495	9.99 980	15
6360	46	8.48 896	411	3-53 634	3-53 614	8.48 917	412	1.51 083	9.99 979	14
6420	47	8.49 304	408	3-53 634	3-53 613	8.49 325	408	1.50 675	9.99 979	13
6480	48	8.49 708	404	3-53 635	3-53 613	8.49 729	404	1.50 271	9.99 979	12
6540	49	8.50 108	400	3-53 635	3-53 613	8.50 130	401	1.49 870	9.99 978	11
6600	50	8.50 504	396	3-53 635	3-53 613	8.50 527	397	1.49 473	9.99 978	10
6660	51	8.50 897	393	3-53 635	3-53 612	8.50 920	393	1.49 080	9.99 977	9
6720	52	8.51 287	390	3-53 635	3-53 612	8.51 310	390	1.48 690	9.99 977	8
6780	53	8.51 673	386	3-53 635	3-53 612	8.51 696	386	1.48 304	9.99 977	7
6840	54	8.52 055	382	3-53 635	3-53 611	8.52 079	383	1.47 921	9.99 976	6
6900	55	8.52 434	379	3-53 635	3-53 611	8.52 459	380	1.47 541	9.99 976	5
6960	56	8.52 810	376	3-53 636	3-53 611	8.52 835	376	1.47 165	9.99 975	4
7020	57	8.53 183	373	3-53 636	3-53 611	8.53 208	373	1.46 792	9.99 975	3
7080	58	8.53 552	369	3-53 636	3-53 610	8.53 578	370	1.46 422	9.99 974	2
7140	59	8.53 919	367	3-53 636	3-53 610	8.53 945	367	1.46 055	9.99 974	1
7200	60	8.54 282	363	3-53 636	3-53 610	8.54 308	363	1.45 692	9.99 974	0
		L. Cos.	d.			L. Cot.	c. d.	L. Tan.	L. Sin.	'

"	'	L. Sin.	d.	Cpl. 8'.	Cpl. T'.	L. Tan.	c. d.	L. Cot.	L. Cos.	
7200	0	8.54 282	360	3.53 636	3.53 610	8.54 308	361	1.45 692	9.99 974	60
7260	1	8.54 642	357	3.53 636	3.53 609	8.54 669	358	1.45 331	9.99 973	59
7320	2	8.54 999	355	3.53 637	3.53 609	8.55 027	355	1.44 973	9.99 973	58
7380	3	8.55 354	351	3.53 637	3.53 609	8.55 382	352	1.44 618	9.99 972	57
7440	4	8.55 705	349	3.53 637	3.53 609	8.55 734	349	1.44 266	9.99 972	56
7500	5	8.56 054	346	3.53 637	3.53 608	8.56 083	346	1.43 917	9.99 971	55
7560	6	8.56 400	343	3.53 637	3.53 608	8.56 429	344	1.43 571	9.99 971	54
7620	7	8.56 743	341	3.53 637	3.53 608	8.56 773	341	1.43 227	9.99 970	53
7680	8	8.57 084	337	3.53 637	3.53 607	8.57 114	338	1.42 886	9.99 970	52
7740	9	8.57 421	336	3.53 638	3.53 607	8.57 452	336	1.42 548	9.99 969	51
7800	10	8.57 757	332	3.53 638	3.53 607	8.57 788	333	1.42 212	9.99 969	50
7860	11	8.58 089	330	3.53 638	3.53 606	8.58 121	330	1.41 879	9.99 968	49
7920	12	8.58 419	328	3.53 638	3.53 606	8.58 451	328	1.41 549	9.99 968	48
7980	13	8.58 747	325	3.53 638	3.53 606	8.58 779	326	1.41 221	9.99 967	47
8040	14	8.59 072	323	3.53 638	3.53 605	8.59 105	323	1.40 895	9.99 967	46
8100	15	8.59 395	320	3.53 639	3.53 605	8.59 428	321	1.40 572	9.99 967	45
8160	16	8.59 715	318	3.53 639	3.53 605	8.59 749	319	1.40 251	9.99 966	44
8220	17	8.60 033	316	3.53 639	3.53 604	8.60 068	316	1.39 932	9.99 966	43
8280	18	8.60 349	313	3.53 639	3.53 604	8.60 384	314	1.39 616	9.99 965	42
8340	19	8.60 662	311	3.53 639	3.53 604	8.60 698	311	1.39 302	9.99 964	41
8400	20	8.60 973	309	3.53 639	3.53 603	8.61 009	310	1.38 991	9.99 964	40
8460	21	8.61 282	307	3.53 640	3.53 603	8.61 319	307	1.38 681	9.99 963	39
8520	22	8.61 589	305	3.53 640	3.53 603	8.61 626	305	1.38 374	9.99 963	38
8580	23	8.61 894	302	3.53 640	3.53 602	8.61 931	303	1.38 069	9.99 962	37
8640	24	8.62 196	301	3.53 640	3.53 602	8.62 234	301	1.37 766	9.99 962	36
8700	25	8.62 497	298	3.53 640	3.53 602	8.62 535	299	1.37 463	9.99 961	35
8760	26	8.62 795	296	3.53 640	3.53 601	8.62 834	297	1.37 166	9.99 961	34
8820	27	8.63 091	294	3.53 641	3.53 601	8.63 131	295	1.36 869	9.99 960	33
8880	28	8.63 385	293	3.53 641	3.53 601	8.63 426	292	1.36 574	9.99 960	32
8940	29	8.63 678	290	3.53 641	3.53 600	8.63 718	291	1.36 282	9.99 959	31
9000	30	8.63 968	288	3.53 641	3.53 600	8.64 009	289	1.35 991	9.99 959	30
9060	31	8.64 256	287	3.53 641	3.53 599	8.64 298	287	1.35 702	9.99 958	29
9120	32	8.64 543	284	3.53 642	3.53 599	8.64 585	285	1.35 415	9.99 958	28
9180	33	8.64 827	283	3.53 642	3.53 599	8.64 870	284	1.35 130	9.99 957	27
9240	34	8.65 110	281	3.53 642	3.53 598	8.65 154	281	1.34 846	9.99 956	26
9300	35	8.65 391	279	3.53 642	3.53 598	8.65 435	280	1.34 565	9.99 956	25
9360	36	8.65 670	277	3.53 642	3.53 598	8.65 715	278	1.34 285	9.99 955	24
9420	37	8.65 947	276	3.53 642	3.53 597	8.65 993	276	1.34 007	9.99 955	23
9480	38	8.66 223	274	3.53 643	3.53 597	8.66 269	274	1.33 731	9.99 954	22
9540	39	8.66 497	272	3.53 643	3.53 596	8.66 543	273	1.33 457	9.99 954	21
9600	40	8.66 769	270	3.53 643	3.53 596	8.66 816	271	1.33 184	9.99 953	20
9660	41	8.67 039	269	3.53 643	3.53 595	8.67 087	269	1.32 913	9.99 952	19
9720	42	8.67 308	267	3.53 644	3.53 595	8.67 356	268	1.32 644	9.99 952	18
9780	43	8.67 575	266	3.53 644	3.53 595	8.67 624	266	1.32 376	9.99 951	17
9840	44	8.67 841	263	3.53 644	3.53 594	8.67 890	264	1.32 110	9.99 951	16
9900	45	8.68 104	263	3.53 644	3.53 594	8.68 154	263	1.31 846	9.99 950	15
9960	46	8.68 367	260	3.53 644	3.53 594	8.68 417	261	1.31 583	9.99 949	14
10020	47	8.68 627	259	3.53 644	3.53 593	8.68 678	260	1.31 322	9.99 949	13
10080	48	8.68 886	258	3.53 645	3.53 593	8.68 938	258	1.31 062	9.99 948	12
10140	49	8.69 144	256	3.53 645	3.53 592	8.69 196	257	1.30 804	9.99 948	11
10200	50	8.69 400	254	3.53 645	3.53 592	8.69 453	255	1.30 547	9.99 947	10
10260	51	8.69 654	253	3.53 646	3.53 591	8.69 708	254	1.30 292	9.99 946	9
10320	52	8.69 907	252	3.53 646	3.53 591	8.69 962	252	1.30 038	9.99 946	8
10380	53	8.70 159	250	3.53 646	3.53 590	8.70 214	251	1.29 786	9.99 945	7
10440	54	8.70 409	249	3.53 646	3.53 590	8.70 465	249	1.29 535	9.99 944	6
10500	55	8.70 658	247	3.53 646	3.53 589	8.70 714	248	1.29 286	9.99 944	5
10560	56	8.70 905	246	3.53 647	3.53 589	8.70 962	246	1.29 038	9.99 943	4
10620	57	8.71 151	244	3.53 647	3.53 589	8.71 208	245	1.28 792	9.99 942	3
10680	58	8.71 395	243	3.53 647	3.53 588	8.71 453	244	1.28 547	9.99 942	2
10740	59	8.71 638	242	3.53 647	3.53 588	8.71 697	243	1.28 303	9.99 941	1
10800	60	8.71 880		3.53 647	3.53 588	8.71 940		1.28 060	9.99 940	0
		L. Cos.	d.			L. Cot.	c. d.	L. Tan.	L. Sin.	'

	L. Sin.	d.	L. Tan.	c.d.	L. Cot.	L. Cos.		P. P.
0	8.71 880		8.71 940		1.28 060	9.99 940	60	
1	8.72 120	240	8.72 181	241	1.27 819	9.99 940	59	
2	8.72 359	239	8.72 420	239	1.27 580	9.99 939	58	241 239 237 236 234
3	8.72 597	238	8.72 659	237	1.27 341	9.99 938	57	1 24.1 23.9 23.7 23.6 23.4
4	8.72 834	237	8.72 896	236	1.27 104	9.99 938	56	2 48.2 47.8 47.4 47.2 46.8
5	8.73 069	235	8.73 132	234	1.26 868	9.99 937	55	3 72.3 71.7 71.1 70.8 70.2
6	8.73 303	234	8.73 366	234	1.26 634	9.99 936	54	4 96.4 95.6 94.8 94.4 93.6
7	8.73 535	232	8.73 600	232	1.26 400	9.99 936	53	5 120.3 119.5 118.5 118.0 117.0
8	8.73 767	230	8.73 832	231	1.26 168	9.99 935	52	6 144.6 143.4 142.2 141.6 140.4
9	8.73 997	229	8.74 063	229	1.25 937	9.99 934	51	7 168.7 167.3 165.9 165.2 163.8
10	8.74 226	228	8.74 292	229	1.25 708	9.99 934	50	8 192.8 191.2 189.6 188.8 187.2
11	8.74 454	226	8.74 521	227	1.25 479	9.99 933	49	9 216.9 215.1 213.3 212.4 210.6
12	8.74 680	226	8.74 748	226	1.25 252	9.99 932	48	
13	8.74 906	224	8.74 974	225	1.25 026	9.99 932	47	1 23.2 23.1 22.9 22.7 22.6
14	8.75 130	223	8.75 199	224	1.24 801	9.99 931	46	2 46.4 46.2 45.8 45.4 45.2
15	8.75 353	222	8.75 423	222	1.24 577	9.99 930	45	3 69.6 69.3 68.7 68.1 67.8
16	8.75 575	220	8.75 645	222	1.24 355	9.99 929	44	4 92.8 92.4 91.6 90.8 90.4
17	8.75 795	220	8.75 867	220	1.24 133	9.99 929	43	5 116.0 115.5 114.5 113.5 113.0
18	8.76 015	219	8.76 087	219	1.23 913	9.99 928	42	6 139.2 138.6 137.4 136.8 135.6
19	8.76 234	217	8.76 306	219	1.23 694	9.99 927	41	7 162.4 161.7 160.3 158.9 158.2
20	8.76 451	216	8.76 523	217	1.23 475	9.99 926	40	8 185.6 184.8 183.2 181.6 180.8
21	8.76 667	216	8.76 742	216	1.23 258	9.99 926	39	9 208.8 207.9 206.1 204.3 203.4
22	8.76 883	214	8.76 958	215	1.23 042	9.99 925	38	
23	8.77 097	213	8.77 173	214	1.22 827	9.99 924	37	1 224 222 220 219 217
24	8.77 310	212	8.77 387	213	1.22 613	9.99 923	36	2 22.4 22.2 22.0 21.9 21.7
25	8.77 522	211	8.77 600	211	1.22 400	9.99 923	35	3 44.8 44.4 44.0 43.8 43.4
26	8.77 733	210	8.77 811	211	1.22 189	9.99 922	34	4 67.2 66.6 66.0 65.7 65.1
27	8.77 943	209	8.78 022	210	1.21 978	9.99 921	33	5 89.6 88.8 88.0 87.6 86.8
28	8.78 152	208	8.78 232	209	1.21 768	9.99 920	32	6 112.0 111.0 110.0 109.5 108.5
29	8.78 360	208	8.78 441	208	1.21 559	9.99 920	31	7 134.4 133.2 132.0 131.4 130.8
30	8.78 568	206	8.78 649	206	1.21 351	9.99 919	30	8 156.8 155.4 154.0 153.3 151.9
31	8.78 774	205	8.78 855	206	1.21 145	9.99 918	29	9 179.2 177.6 176.0 175.2 173.6
32	8.78 979	204	8.79 061	205	1.20 939	9.99 917	28	
33	8.79 183	203	8.79 266	204	1.20 734	9.99 917	27	1 216 214 213 211 209
34	8.79 386	202	8.79 470	203	1.20 530	9.99 916	26	2 21.6 21.4 21.3 21.1 20.9
35	8.79 588	201	8.79 673	202	1.20 327	9.99 915	25	3 43.2 42.8 42.6 42.2 41.8
36	8.79 789	201	8.79 875	201	1.20 125	9.99 914	24	4 64.8 64.2 63.9 63.3 62.7
37	8.79 990	199	8.80 076	201	1.19 924	9.99 913	23	5 86.4 85.6 85.2 84.4 83.6
38	8.80 189	199	8.80 277	199	1.19 723	9.99 913	22	6 108.0 107.0 106.5 105.5 104.5
39	8.80 388	197	8.80 476	198	1.19 524	9.99 912	21	7 129.6 128.4 127.8 126.6 125.4
40	8.80 585	197	8.80 674	198	1.19 326	9.99 911	20	8 151.2 149.8 149.1 147.7 146.3
41	8.80 782	196	8.80 872	196	1.19 128	9.99 910	19	9 172.8 171.2 170.4 168.8 167.2
42	8.80 978	195	8.81 068	196	1.18 932	9.99 909	18	
43	8.81 173	194	8.81 264	195	1.18 736	9.99 909	17	1 194.4 192.6 191.7 189.9 188.1
44	8.81 367	193	8.81 459	194	1.18 541	9.99 908	16	
45	8.81 560	192	8.81 653	193	1.18 347	9.99 907	15	2 208 206 203 201 199
46	8.81 752	192	8.81 846	192	1.18 154	9.99 906	14	3 20.8 20.6 20.3 20.1 19.9
47	8.81 944	190	8.82 038	192	1.17 962	9.99 905	13	4 41.6 41.2 40.6 40.2 39.8
48	8.82 134	190	8.82 230	190	1.17 770	9.99 904	12	5 62.4 61.8 60.9 60.3 59.7
49	8.82 324	189	8.82 420	190	1.17 580	9.99 904	11	6 83.2 82.4 81.2 80.4 79.6
50	8.82 513	188	8.82 610	189	1.17 390	9.99 903	10	7 104.0 103.0 101.5 100.5 99.5
51	8.82 701	187	8.82 799	188	1.17 201	9.99 902	9	8 124.8 123.6 121.8 120.6 119.4
52	8.82 888	187	8.82 987	188	1.17 013	9.99 901	8	9 145.6 144.2 142.1 140.7 139.3
53	8.83 075	186	8.83 175	186	1.16 825	9.99 900	7	8 166.4 164.8 162.4 160.8 159.2
54	8.83 261	185	8.83 361	186	1.16 639	9.99 899	6	9 187.2 185.4 182.7 180.9 179.1
55	8.83 446	184	8.83 547	185	1.16 453	9.99 898	5	
56	8.83 630	183	8.83 732	184	1.16 268	9.99 898	4	1 198 196 194 192 190
57	8.83 813	183	8.83 916	184	1.16 084	9.99 897	3	2 19.8 19.6 19.4 19.2 19.0
58	8.83 996	181	8.84 100	182	1.15 900	9.99 896	2	3 39.6 39.2 38.8 38.4 38.0
59	8.84 177	181	8.84 282	182	1.15 718	9.99 895	1	4 59.4 58.8 58.2 57.6 57.0
60	8.84 358		8.84 464		1.15 536	9.99 894	0	5 79.2 78.4 77.6 76.8 76.0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.		6 99.0 98.0 97.0 96.0 95.0
								7 118.8 117.6 116.4 115.2 114.0
								8 138.6 137.2 135.8 134.4 133.0
								9 158.4 156.8 155.2 153.6 152.0
								1 178.2 176.4 174.6 172.8 171.0
								2 188 186 184 182 181
								3 18.8 18.6 18.4 18.2 18.1
								4 37.6 37.2 36.8 36.4 36.2
								5 56.4 55.8 55.2 54.6 54.3
								6 75.2 74.4 73.6 72.8 72.4
								7 94.0 93.0 92.0 91.0 90.5
								8 112.8 111.6 110.4 109.2 108.6
								9 131.6 130.2 128.8 127.4 126.7
								1 150.4 148.8 147.2 145.6 144.8
								2 169.2 167.4 165.6 163.8 162.9
								3 188 186 184 182 181
								4 18.8 18.6 18.4 18.2 18.1
								5 37.6 37.2 36.8 36.4 36.2
								6 56.4 55.8 55.2 54.6 54.3
								7 75.2 74.4 73.6 72.8 72.4
								8 94.0 93.0 92.0 91.0 90.5
								9 112.8 111.6 110.4 109.2 108.6
								1 131.6 130.2 128.8 127.4 126.7
								2 150.4 148.8 147.2 145.6 144.8
								3 169.2 167.4 165.6 163.8 162.9
								4 188 186 184 182 181
								5 18.8 18.6 18.4 18.2 18.1
								6 37.6 37.2 36.8 36.4 36.2
								7 56.4 55.8 55.2 54.6 54.3
								8 75.2 74.4 73.6 72.8 72.4
								9 94.0 93.0 92.0 91.0 90.5
								1 112.8 111.6 110.4 109.2 108.6
								2 131.6 130.2 128.8 127.4 126.7
								3 150.4 148.8 147.2 145.6 144.8
								4 169.2 167.4 165.6 163.8 162.9

	L. Sin.	d.	L. Tan.	c.d.	L. Cot.	L. Cos.		P. P.
0	8.94 030		8.94 195		1.05 805	9.99 834	60	
1	8.94 174	144	8.94 340	145	1.05 660	9.99 833	59	
2	8.94 317	143	8.94 485	145	1.05 515	9.99 832	58	
3	8.94 461	144	8.94 630	145	1.05 370	9.99 831	57	
4	8.94 603	142	8.94 773	143	1.05 227	9.99 830	56	
5	8.94 746	143	8.94 917	144	1.05 083	9.99 829	55	
6	8.94 887	141	8.95 060	143	1.04 940	9.99 828	54	
7	8.95 029	142	8.95 202	142	1.04 798	9.99 827	53	
8	8.95 170	141	8.95 344	142	1.04 656	9.99 825	52	
9	8.95 310	140	8.95 486	142	1.04 514	9.99 824	51	
10	8.95 450	140	8.95 627	141	1.04 373	9.99 823	50	
11	8.95 589	139	8.95 767	140	1.04 233	9.99 822	49	
12	8.95 728	139	8.95 908	141	1.04 092	9.99 821	48	
13	8.95 867	139	8.96 047	139	1.03 953	9.99 820	47	
14	8.96 005	138	8.96 187	140	1.03 813	9.99 819	46	
15	8.96 143	138	8.96 325	138	1.03 675	9.99 817	45	
16	8.96 280	137	8.96 464	139	1.03 536	9.99 816	44	
17	8.96 417	137	8.96 602	138	1.03 398	9.99 815	43	
18	8.96 553	136	8.96 739	137	1.03 261	9.99 814	42	
19	8.96 689	136	8.96 877	138	1.03 123	9.99 813	41	
20	8.96 825	136	8.97 013	136	1.02 987	9.99 812	40	
21	8.96 960	135	8.97 150	137	1.02 850	9.99 810	39	
22	8.97 095	135	8.97 285	135	1.02 715	9.99 809	38	
23	8.97 229	134	8.97 421	136	1.02 579	9.99 808	37	
24	8.97 363	134	8.97 556	135	1.02 444	9.99 807	36	
25	8.97 496	133	8.97 691	135	1.02 309	9.99 806	35	
26	8.97 629	133	8.97 825	134	1.02 175	9.99 804	34	
27	8.97 762	133	8.97 959	134	1.02 041	9.99 803	33	
28	8.97 894	132	8.98 092	133	1.01 908	9.99 802	32	
29	8.98 026	132	8.98 225	133	1.01 775	9.99 801	31	
30	8.98 157	131	8.98 358	133	1.01 642	9.99 800	30	
31	8.98 288	131	8.98 490	132	1.01 510	9.99 798	29	
32	8.98 419	131	8.98 622	132	1.01 378	9.99 797	28	
33	8.98 549	130	8.98 753	131	1.01 247	9.99 796	27	
34	8.98 679	130	8.98 884	131	1.01 116	9.99 795	26	
35	8.98 808	129	8.99 015	131	1.00 985	9.99 793	25	
36	8.98 937	129	8.99 145	130	1.00 855	9.99 792	24	
37	8.99 066	129	8.99 275	130	1.00 725	9.99 791	23	
38	8.99 194	128	8.99 405	130	1.00 595	9.99 790	22	
39	8.99 322	128	8.99 534	129	1.00 466	9.99 788	21	
40	8.99 450	128	8.99 662	128	1.00 338	9.99 787	20	
41	8.99 577	127	8.99 791	129	1.00 209	9.99 786	19	
42	8.99 704	127	8.99 919	128	1.00 081	9.99 785	18	
43	8.99 830	126	9.00 046	127	0.99 954	9.99 783	17	
44	8.99 956	126	9.00 174	128	0.99 826	9.99 782	16	
45	9.00 082	126	9.00 301	127	0.99 699	9.99 781	15	
46	9.00 207	125	9.00 427	126	0.99 573	9.99 780	14	
47	9.00 332	125	9.00 553	126	0.99 447	9.99 778	13	
48	9.00 456	124	9.00 679	126	0.99 321	9.99 777	12	
49	9.00 581	125	9.00 805	126	0.99 195	9.99 776	11	
50	9.00 704	123	9.00 930	125	0.99 070	9.99 775	10	
51	9.00 828	124	9.01 055	125	0.98 945	9.99 773	9	
52	9.00 951	123	9.01 179	124	0.98 821	9.99 772	8	
53	9.01 074	123	9.01 303	124	0.98 697	9.99 771	7	
54	9.01 196	122	9.01 427	124	0.98 573	9.99 769	6	
55	9.01 318	122	9.01 550	123	0.98 450	9.99 768	5	
56	9.01 440	122	9.01 673	123	0.98 327	9.99 767	4	
57	9.01 561	121	9.01 796	123	0.98 204	9.99 765	3	
58	9.01 682	121	9.01 918	122	0.98 082	9.99 764	2	
59	9.01 803	121	9.02 040	122	0.97 960	9.99 763	1	
60	9.01 923	120	9.02 162	122	0.97 838	9.99 761	0	
	L. Cos.	d.	L. Cot.	c.d.	L. Tan.	L. Sin.		P. P.

	L. Sin.	d.	L. Tan.	c.d.	L. Cot.	L. Cos.	P. P.
0	9.01 923		9.02 162		0.97 838	9.90 761	60
1	9.02 043	120	9.02 283	121	0.97 717	9.99 760	59
2	9.02 163	120	9.02 404	121	0.97 596	9.99 759	58
3	9.02 283	120	9.02 525	121	0.97 475	9.99 757	57
4	9.02 402	119	9.02 645	120	0.97 355	9.99 756	56
5	9.02 520	118	9.02 766	121	0.97 234	9.99 755	55
6	9.02 639	119	9.02 885	119	0.97 115	9.99 753	54
7	9.02 757	118	9.03 005	120	0.96 995	9.99 752	53
8	9.02 874	117	9.03 124	119	0.96 876	9.99 751	52
9	9.02 992	118	9.03 242	118	0.96 758	9.99 749	51
10	9.03 109	117	9.03 361	119	0.96 639	9.99 748	50
11	9.03 226	117	9.03 479	118	0.96 521	9.99 747	49
12	9.03 342	116	9.03 597	118	0.96 403	9.99 745	48
13	9.03 458	116	9.03 714	117	0.96 286	9.99 744	47
14	9.03 574	116	9.03 832	118	0.96 168	9.99 742	46
15	9.03 690	116	9.03 948	116	0.96 052	9.99 741	45
16	9.03 805	115	9.04 065	117	0.95 935	9.99 740	44
17	9.03 920	115	9.04 181	116	0.95 819	9.99 738	43
18	9.04 034	114	9.04 297	116	0.95 703	9.99 737	42
19	9.04 149	115	9.04 413	116	0.95 587	9.99 736	41
20	9.04 262	113	9.04 528	115	0.95 472	9.99 734	40
21	9.04 376	114	9.04 643	115	0.95 357	9.99 733	39
22	9.04 490	114	9.04 758	115	0.95 242	9.99 731	38
23	9.04 603	113	9.04 873	115	0.95 127	9.99 730	37
24	9.04 715	112	9.04 987	114	0.95 013	9.99 728	36
25	9.04 828	113	9.05 101	114	0.94 899	9.99 727	35
26	9.04 940	112	9.05 214	113	0.94 786	9.99 726	34
27	9.05 052	112	9.05 328	114	0.94 672	9.99 724	33
28	9.05 164	112	9.05 441	113	0.94 559	9.99 723	32
29	9.05 275	111	9.05 553	112	0.94 447	9.99 721	31
30	9.05 386	111	9.05 666	113	0.94 334	9.99 720	30
31	9.05 497	110	9.05 778	112	0.94 222	9.99 718	29
32	9.05 607	110	9.05 890	112	0.94 110	9.99 717	28
33	9.05 717	110	9.06 002	111	0.93 998	9.99 716	27
34	9.05 827	110	9.06 113	111	0.93 887	9.99 714	26
35	9.05 937	110	9.06 224	111	0.93 776	9.99 713	25
36	9.06 046	109	9.06 335	111	0.93 665	9.99 711	24
37	9.06 155	109	9.06 445	110	0.93 555	9.99 710	23
38	9.06 264	109	9.06 556	111	0.93 444	9.99 708	22
39	9.06 372	108	9.06 666	110	0.93 334	9.99 707	21
40	9.06 481	109	9.06 775	109	0.93 225	9.99 705	20
41	9.06 589	108	9.06 885	110	0.93 115	9.99 704	19
42	9.06 696	107	9.06 994	109	0.93 006	9.99 702	18
43	9.06 804	108	9.07 103	109	0.92 897	9.99 701	17
44	9.06 911	107	9.07 211	108	0.92 789	9.99 699	16
45	9.07 018	107	9.07 320	109	0.92 680	9.99 698	15
46	9.07 124	106	9.07 428	108	0.92 572	9.99 696	14
47	9.07 231	107	9.07 536	108	0.92 464	9.99 695	13
48	9.07 337	106	9.07 643	107	0.92 357	9.99 693	12
49	9.07 442	105	9.07 751	108	0.92 249	9.99 692	11
50	9.07 548	106	9.07 858	107	0.92 142	9.99 690	10
51	9.07 653	105	9.07 964	106	0.92 036	9.99 689	9
52	9.07 758	105	9.08 071	107	0.91 929	9.99 687	8
53	9.07 863	105	9.08 177	106	0.91 823	9.99 686	7
54	9.07 968	105	9.08 283	106	0.91 717	9.99 684	6
55	9.08 072	104	9.08 389	106	0.91 611	9.99 683	5
56	9.08 176	104	9.08 495	106	0.91 505	9.99 681	4
57	9.08 280	104	9.08 600	105	0.91 400	9.99 680	3
58	9.08 383	103	9.08 705	105	0.91 295	9.99 678	2
59	9.08 486	103	9.08 810	105	0.91 190	9.99 677	1
60	9.08 589	103	9.08 914	104	0.91 086	9.99 675	0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	P. P.

	L. Sin.	d.	L. Tan.	c.d.	L. Cot.	L. Cos.		P. P.		
0	9.08 589		9.08 914		0.91 086	9.99 675	60			
1	9.08 692	103	9.09 019	105	0.90 981	9.99 674	59			
2	9.08 795	103	9.09 123	104	0.90 877	9.99 672	58			
3	9.08 897	102	9.09 227	104	0.90 773	9.99 670	57		105	104 103
4	9.08 999	102	9.09 330	103	0.90 670	9.99 669	56	1	10.5	10.4 10.3
5	9.09 101	102	9.09 434	104	0.90 566	9.99 667	55	2	21.0	20.8 20.6
6	9.09 202	101	9.09 537	103	0.90 463	9.99 666	54	3	31.5	31.2 30.9
7	9.09 304	102	9.09 640	103	0.90 360	9.99 664	53	4	42.0	41.6 41.2
8	9.09 405	101	9.09 742	102	0.90 258	9.99 663	52	5	52.5	52.0 51.5
9	9.09 506	101	9.09 845	103	0.90 155	9.99 661	51	6	63.0	62.4 61.8
10	9.09 606	100	9.09 947	102	0.90 053	9.99 659	50	7	73.5	72.8 72.1
11	9.09 707	101	9.10 049	102	0.89 951	9.99 658	49	8	84.0	83.2 82.4
12	9.09 807	100	9.10 150	101	0.89 850	9.99 656	48	9	94.5	93.6 92.7
13	9.09 907	100	9.10 252	102	0.89 748	9.99 655	47			
14	9.10 006	99	9.10 353	101	0.89 647	9.99 653	46		102	101 99
15	9.10 106	100	9.10 454	101	0.89 546	9.99 651	45	1	10.2	10.1 9.9
16	9.10 205	99	9.10 555	101	0.89 445	9.99 650	44	2	20.4	20.2 19.8
17	9.10 304	99	9.10 656	101	0.89 344	9.99 648	43	3	30.6	30.3 29.7
18	9.10 402	98	9.10 756	100	0.89 244	9.99 647	42	4	40.8	40.4 39.6
19	9.10 501	99	9.10 856	100	0.89 144	9.99 645	41	5	51.0	50.5 49.5
20	9.10 599	98	9.10 956	100	0.89 044	9.99 643	40	6	61.2	60.6 59.4
21	9.10 697	98	9.11 056	99	0.88 944	9.99 642	39	7	71.4	70.7 69.3
22	9.10 795	98	9.11 155	99	0.88 843	9.99 640	38	8	81.6	80.8 79.2
23	9.10 893	98	9.11 254	99	0.88 746	9.99 638	37	9	91.8	90.9 89.1
24	9.10 990	97	9.11 353	99	0.88 647	9.99 637	36			
25	9.11 087	97	9.11 452	99	0.88 548	9.99 635	35		98	97 96
26	9.11 184	97	9.11 551	99	0.88 449	9.99 633	34	1	9.8	9.7 9.6
27	9.11 281	97	9.11 649	98	0.88 351	9.99 632	33	2	19.6	19.4 19.2
28	9.11 377	96	9.11 747	98	0.88 253	9.99 630	32	3	29.4	29.1 28.8
29	9.11 474	96	9.11 845	98	0.88 153	9.99 629	31	4	39.2	38.8 38.4
30	9.11 570	96	9.11 943	97	0.88 057	9.99 627	30	5	49.0	48.5 48.0
31	9.11 666	95	9.12 040	98	0.87 960	9.99 625	29	6	58.8	58.2 57.6
32	9.11 761	95	9.12 138	97	0.87 862	9.99 624	28	7	68.6	67.9 67.2
33	9.11 857	95	9.12 235	97	0.87 765	9.99 622	27	8	78.4	77.6 76.8
34	9.11 952	95	9.12 332	96	0.87 668	9.99 620	26	9	88.2	87.3 86.4
35	9.12 047	95	9.12 428	97	0.87 572	9.99 618	25			
36	9.12 142	95	9.12 525	97	0.87 475	9.99 617	24			
37	9.12 236	94	9.12 621	96	0.87 379	9.99 615	23		95	94 93
38	9.12 331	95	9.12 717	96	0.87 283	9.99 613	22	1	9.5	9.4 9.3
39	9.12 425	94	9.12 813	96	0.87 187	9.99 612	21	2	19.0	18.8 18.6
40	9.12 519	94	9.12 909	95	0.87 091	9.99 610	20	3	28.5	28.2 27.9
41	9.12 612	93	9.13 004	95	0.86 996	9.99 608	19	4	38.0	37.6 37.2
42	9.12 706	94	9.13 099	95	0.86 901	9.99 607	18	5	47.5	47.0 46.5
43	9.12 799	93	9.13 194	95	0.86 806	9.99 605	17	6	57.0	56.4 55.8
44	9.12 892	93	9.13 289	95	0.86 711	9.99 603	16	7	66.5	65.8 65.1
45	9.12 985	93	9.13 384	95	0.86 616	9.99 601	15	8	76.0	75.2 74.4
46	9.13 078	93	9.13 478	94	0.86 522	9.99 600	14	9	85.5	84.6 83.7
47	9.13 171	93	9.13 573	95	0.86 427	9.99 598	13			
48	9.13 263	92	9.13 667	94	0.86 333	9.99 596	12			
49	9.13 355	92	9.13 761	94	0.86 239	9.99 595	11		92	91 90
50	9.13 447	92	9.13 854	93	0.86 146	9.99 593	10	1	9.2	9.1 9.0
51	9.13 539	91	9.13 948	94	0.86 052	9.99 591	9	2	18.4	18.2 18.0
52	9.13 630	91	9.14 041	93	0.85 959	9.99 589	8	3	27.6	27.3 27.0
53	9.13 722	92	9.14 134	93	0.85 866	9.99 588	7	4	36.8	36.4 36.0
54	9.13 813	91	9.14 227	93	0.85 773	9.99 586	6	5	46.0	45.5 45.0
55	9.13 904	91	9.14 320	92	0.85 680	9.99 584	5	6	55.2	54.6 54.0
56	9.13 994	90	9.14 412	92	0.85 588	9.99 582	4	7	64.4	63.7 63.0
57	9.14 085	91	9.14 504	92	0.85 496	9.99 581	3	8	73.6	72.8 72.0
58	9.14 175	90	9.14 597	93	0.85 403	9.99 579	2	9	82.8	81.9 81.0
59	9.14 266	91	9.14 688	91	0.85 312	9.99 577	1			
60	9.14 356	90	9.14 780	92	0.85 220	9.99 575	0			
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.		P. P.		

	L. Sin.	d.	L. Tan.	c.d.	L. Cot.	L. Cos.		P. P.		
0	9.14 356	89	9.14 780	92	0.85 220	9.99 575	60			
1	9.14 445	90	9.14 872	91	0.85 128	9.99 574	59			
2	9.14 535	89	9.14 963	91	0.85 037	9.99 572	58			
3	9.14 624	90	9.15 054	91	0.84 946	9.99 570	57			
4	9.14 714	89	9.15 145	91	0.84 855	9.99 568	56			
5	9.14 803	88	9.15 236	91	0.84 764	9.99 566	55			
6	9.14 891	89	9.15 327	90	0.84 673	9.99 565	54			
7	9.14 980	89	9.15 417	91	0.84 583	9.99 563	53			
8	9.15 069	88	9.15 508	90	0.84 492	9.99 561	52			
9	9.15 157	88	9.15 598	90	0.84 402	9.99 559	51			
10	9.15 245	88	9.15 688	89	0.84 312	9.99 557	50			
11	9.15 333	88	9.15 777	90	0.84 223	9.99 556	49			
12	9.15 421	87	9.15 867	89	0.84 133	9.99 554	48			
13	9.15 508	88	9.15 956	90	0.84 044	9.99 552	47			
14	9.15 596	87	9.16 046	89	0.83 954	9.99 550	46			
15	9.15 683	87	9.16 135	89	0.83 865	9.99 548	45			
16	9.15 770	87	9.16 224	88	0.83 776	9.99 546	44			
17	9.15 857	87	9.16 312	89	0.83 688	9.99 545	43			
18	9.15 944	86	9.16 401	88	0.83 599	9.99 543	42			
19	9.16 030	86	9.16 489	88	0.83 511	9.99 541	41			
20	9.16 116	87	9.16 577	88	0.83 423	9.99 539	40			
21	9.16 203	86	9.16 665	88	0.83 335	9.99 537	39			
22	9.16 289	85	9.16 753	88	0.83 247	9.99 535	38			
23	9.16 374	86	9.16 841	87	0.83 159	9.99 533	37			
24	9.16 460	85	9.16 928	88	0.83 072	9.99 532	36			
25	9.16 545	86	9.17 016	87	0.82 984	9.99 530	35			
26	9.16 631	85	9.17 103	87	0.82 897	9.99 528	34			
27	9.16 716	85	9.17 190	87	0.82 810	9.99 526	33			
28	9.16 801	85	9.17 277	86	0.82 723	9.99 524	32			
29	9.16 886	84	9.17 363	87	0.82 637	9.99 522	31			
30	9.16 970	85	9.17 450	86	0.82 550	9.99 520	30			
31	9.17 055	84	9.17 536	86	0.82 464	9.99 518	29			
32	9.17 139	84	9.17 622	86	0.82 378	9.99 517	28			
33	9.17 223	84	9.17 708	86	0.82 292	9.99 515	27			
34	9.17 307	84	9.17 794	86	0.82 206	9.99 513	26			
35	9.17 391	83	9.17 880	85	0.82 120	9.99 511	25			
36	9.17 474	84	9.17 965	86	0.82 035	9.99 509	24			
37	9.17 558	83	9.18 051	85	0.81 949	9.99 507	23			
38	9.17 641	83	9.18 136	85	0.81 864	9.99 505	22			
39	9.17 724	83	9.18 221	85	0.81 779	9.99 503	21			
40	9.17 807	83	9.18 306	85	0.81 694	9.99 501	20			
41	9.17 890	83	9.18 391	84	0.81 609	9.99 499	19			
42	9.17 973	82	9.18 475	85	0.81 525	9.99 497	18			
43	9.18 055	82	9.18 560	84	0.81 440	9.99 495	17			
44	9.18 137	83	9.18 644	84	0.81 356	9.99 494	16			
45	9.18 220	82	9.18 728	84	0.81 272	9.99 492	15			
46	9.18 302	81	9.18 812	84	0.81 188	9.99 490	14			
47	9.18 383	82	9.18 896	83	0.81 104	9.99 488	13			
48	9.18 465	82	9.18 979	84	0.81 021	9.99 486	12			
49	9.18 547	81	9.19 063	83	0.80 937	9.99 484	11			
50	9.18 628	81	9.19 146	83	0.80 854	9.99 482	10			
51	9.18 709	81	9.19 229	83	0.80 771	9.99 480	9			
52	9.18 790	81	9.19 312	83	0.80 688	9.99 478	8			
53	9.18 871	81	9.19 395	83	0.80 605	9.99 476	7			
54	9.18 952	81	9.19 478	83	0.80 522	9.99 474	6			
55	9.19 033	80	9.19 561	82	0.80 439	9.99 472	5			
56	9.19 113	80	9.19 643	82	0.80 357	9.99 470	4			
57	9.19 193	80	9.19 725	82	0.80 275	9.99 468	3			
58	9.19 273	80	9.19 807	82	0.80 193	9.99 466	2			
59	9.19 353	80	9.19 889	82	0.80 111	9.99 464	1			
60	9.19 433		9.19 971		0.80 029	9.99 462	0			
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.		P. P.		

	92	91	90
1	9.2	9.1	9.0
2	18.4	18.2	18.0
3	27.6	27.3	27.0
4	36.8	36.4	36.0
5	46.0	45.5	45.0
6	55.2	54.6	54.0
7	64.4	63.7	63.0
8	73.6	72.8	72.0
9	82.8	81.9	81.0

	89	88
1	8.9	8.8
2	17.8	17.6
3	26.7	26.4
4	35.6	35.2
5	44.5	44.0
6	53.4	52.8
7	62.3	61.6
8	71.2	70.4
9	80.1	79.2

	87	86	85
1	8.7	8.6	8.5
2	17.4	17.2	17.0
3	26.1	25.8	25.5
4	34.8	34.4	34.0
5	43.5	43.0	42.5
6	52.2	51.6	51.0
7	60.9	60.2	59.5
8	69.6	68.8	68.0
9	78.3	77.4	76.5

	84	83
1	8.4	8.3
2	16.8	16.6
3	25.2	24.9
4	33.6	33.2
5	42.0	41.5
6	50.4	49.8
7	58.8	58.1
8	67.2	66.4
9	75.6	74.7

	82	81	80
1	8.2	8.1	8.0
2	16.4	16.2	16.0
3	24.6	24.3	24.0
4	32.8	32.4	32.0
5	41.0	40.5	40.0
6	49.2	48.6	48.0
7	57.4	56.7	56.0
8	65.6	64.8	64.0
9	73.8	72.9	72.0

	L. Sin.	d.	L. Tan.	c.d.	L. Cot.	L. Cos.		P. P.
0	9.19 433	80	9.19 971	82	0.80 029	9.99 462	60	
1	9.19 513	79	9.20 053	81	0.79 947	9.99 460	59	
2	9.19 592	80	9.20 134	82	0.79 866	9.99 458	58	
3	9.19 672	79	9.20 216	81	0.79 784	9.99 456	57	
4	9.19 751	79	9.20 297	81	0.79 703	9.99 454	56	
5	9.19 830	79	9.20 378	81	0.79 622	9.99 452	55	
6	9.19 909	79	9.20 459	81	0.79 541	9.99 450	54	
7	9.19 988	79	9.20 540	81	0.79 460	9.99 448	53	
8	9.20 067	78	9.20 621	80	0.79 379	9.99 446	52	
9	9.20 145	78	9.20 701	81	0.79 299	9.99 444	51	
10	9.20 223	79	9.20 782	80	0.79 218	9.99 442	50	
11	9.20 302	78	9.20 862	80	0.79 138	9.99 440	49	
12	9.20 380	78	9.20 942	80	0.79 058	9.99 438	48	
13	9.20 458	77	9.21 022	80	0.78 978	9.99 436	47	
14	9.20 535	78	9.21 102	80	0.78 898	9.99 434	46	
15	9.20 613	78	9.21 182	79	0.78 818	9.99 432	45	
16	9.20 691	77	9.21 261	80	0.78 739	9.99 429	44	
17	9.20 768	77	9.21 341	79	0.78 659	9.99 427	43	
18	9.20 845	77	9.21 420	79	0.78 580	9.99 425	42	
19	9.20 922	77	9.21 499	79	0.78 501	9.99 423	41	
20	9.20 999	77	9.21 578	79	0.78 422	9.99 421	40	
21	9.21 076	77	9.21 657	79	0.78 343	9.99 419	39	
22	9.21 153	76	9.21 736	78	0.78 264	9.99 417	38	
23	9.21 229	77	9.21 814	79	0.78 186	9.99 415	37	
24	9.21 306	76	9.21 893	78	0.78 107	9.99 413	36	
25	9.21 382	76	9.21 971	78	0.78 029	9.99 411	35	
26	9.21 458	76	9.22 049	78	0.77 951	9.99 409	34	
27	9.21 534	76	9.22 127	78	0.77 873	9.99 407	33	
28	9.21 610	75	9.22 205	78	0.77 795	9.99 404	32	
29	9.21 685	76	9.22 283	78	0.77 717	9.99 402	31	
30	9.21 761	75	9.22 361	77	0.77 639	9.99 400	30	
31	9.21 836	76	9.22 438	78	0.77 562	9.99 398	29	
32	9.21 912	75	9.22 516	77	0.77 484	9.99 396	28	
33	9.21 987	75	9.22 593	77	0.77 407	9.99 394	27	
34	9.22 062	75	9.22 670	77	0.77 330	9.99 392	26	
35	9.22 137	74	9.22 747	77	0.77 253	9.99 390	25	
36	9.22 211	75	9.22 824	77	0.77 176	9.99 388	24	
37	9.22 286	75	9.22 901	77	0.77 099	9.99 385	23	
38	9.22 361	75	9.22 977	76	0.77 023	9.99 383	22	
39	9.22 435	74	9.23 054	77	0.76 946	9.99 381	21	
40	9.22 509	74	9.23 130	76	0.76 870	9.99 379	20	
41	9.22 583	74	9.23 206	77	0.76 794	9.99 377	19	
42	9.22 657	74	9.23 283	77	0.76 717	9.99 375	18	
43	9.22 731	74	9.23 359	76	0.76 641	9.99 372	17	
44	9.22 805	74	9.23 435	76	0.76 565	9.99 370	16	
45	9.22 878	73	9.23 510	75	0.76 490	9.99 368	15	
46	9.22 952	74	9.23 586	76	0.76 414	9.99 366	14	
47	9.23 025	73	9.23 661	75	0.76 339	9.99 364	13	
48	9.23 098	73	9.23 737	76	0.76 263	9.99 362	12	
49	9.23 171	73	9.23 812	75	0.76 188	9.99 359	11	
50	9.23 244	73	9.23 887	75	0.76 113	9.99 357	10	
51	9.23 317	73	9.23 962	75	0.76 038	9.99 355	9	
52	9.23 390	72	9.24 037	75	0.75 963	9.99 353	8	
53	9.23 462	72	9.24 112	75	0.75 888	9.99 351	7	
54	9.23 535	73	9.24 186	74	0.75 814	9.99 348	6	
55	9.23 607	72	9.24 261	75	0.75 739	9.99 346	5	
56	9.23 679	72	9.24 335	74	0.75 665	9.99 344	4	
57	9.23 752	73	9.24 410	75	0.75 590	9.99 342	3	
58	9.23 823	71	9.24 484	74	0.75 516	9.99 340	2	
59	9.23 895	72	9.24 558	74	0.75 442	9.99 337	1	
60	9.23 967	72	9.24 632	74	0.75 368	9.99 335	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.23 967	72	9.24 632	74	0.75 368	9.99 335	2	60	
1	9.24 039	71	9.24 706	73	0.75 294	9.99 333	2	59	74 73 72
2	9.24 110	71	9.24 779	74	0.75 221	9.99 331	3	58	
3	9.24 181	71	9.24 853	73	0.75 147	9.99 328	2	57	1 7.4 7.3 7.2
4	9.24 253	72	9.24 926	74	0.75 074	9.99 326	2	56	2 14.8 14.6 14.4
5	9.24 324	71	9.25 000	73	0.75 000	9.99 324	2	55	3 22.2 21.9 21.6
6	9.24 395	71	9.25 073	73	0.74 927	9.99 322	2	54	4 29.6 29.2 28.8
7	9.24 466	70	9.25 146	73	0.74 854	9.99 319	3	53	5 37.0 36.5 36.0
8	9.24 536	71	9.25 219	73	0.74 781	9.99 317	2	52	6 44.4 43.8 43.2
9	9.24 607	70	9.25 292	73	0.74 708	9.99 315	2	51	7 51.8 51.1 50.4
10	9.24 677	71	9.25 365	72	0.74 635	9.99 313	3	50	8 59.2 58.4 57.6
11	9.24 748	70	9.25 437	73	0.74 563	9.99 310	2	49	9 66.6 65.7 64.8
12	9.24 818	70	9.25 510	72	0.74 490	9.99 308	2	48	
13	9.24 888	70	9.25 582	73	0.74 418	9.99 306	2	47	71 70 69
14	9.24 958	70	9.25 655	72	0.74 345	9.99 304	3	46	1 7.1 7.0 6.9
15	9.25 028	70	9.25 727	72	0.74 273	9.99 301	2	45	2 14.2 14.0 13.8
16	9.25 098	70	9.25 799	72	0.74 201	9.99 299	2	44	3 21.3 21.0 20.7
17	9.25 168	69	9.25 871	72	0.74 129	9.99 297	3	43	4 28.4 28.0 27.6
18	9.25 237	70	9.25 943	72	0.74 057	9.99 294	2	42	5 35.5 35.0 34.5
19	9.25 307	69	9.26 015	71	0.73 985	9.99 292	2	41	6 42.6 42.0 41.4
20	9.25 376	69	9.26 086	72	0.73 914	9.99 290	2	40	7 49.7 49.0 48.3
21	9.25 445	69	9.26 158	71	0.73 842	9.99 288	3	39	8 56.8 56.0 55.2
22	9.25 514	69	9.26 229	72	0.73 771	9.99 285	2	38	9 63.9 63.0 62.1
23	9.25 583	69	9.26 301	71	0.73 699	9.99 283	2	37	
24	9.25 652	69	9.26 372	71	0.73 628	9.99 281	3	36	68 67 66
25	9.25 721	69	9.26 443	71	0.73 557	9.99 278	2	35	1 6.8 6.7 6.6
26	9.25 790	68	9.26 514	71	0.73 486	9.99 276	2	34	2 13.6 13.4 13.2
27	9.25 858	69	9.26 585	70	0.73 415	9.99 274	3	33	3 20.4 20.1 19.8
28	9.25 927	68	9.26 655	71	0.73 345	9.99 271	2	32	4 27.2 26.8 26.4
29	9.25 995	68	9.26 726	71	0.73 274	9.99 269	2	31	5 34.0 33.5 33.0
30	9.26 063	68	9.26 797	70	0.73 203	9.99 267	3	30	6 40.8 40.2 39.6
31	9.26 131	68	9.26 867	70	0.73 133	9.99 264	2	29	7 47.6 46.9 46.2
32	9.26 199	68	9.26 937	71	0.73 063	9.99 262	2	28	8 54.4 53.6 52.8
33	9.26 267	68	9.27 008	70	0.72 992	9.99 260	3	27	9 61.2 60.3 59.4
34	9.26 335	68	9.27 078	70	0.72 922	9.99 257	2	26	
35	9.26 403	67	9.27 148	70	0.72 852	9.99 255	2	25	85 3
36	9.26 470	68	9.27 218	70	0.72 782	9.99 252	3	24	1 6.5 0.3
37	9.26 538	67	9.27 288	69	0.72 712	9.99 250	2	23	2 13.0 0.6
38	9.26 605	67	9.27 357	70	0.72 643	9.99 248	2	22	3 19.5 0.9
39	9.26 672	67	9.27 427	69	0.72 573	9.99 245	3	21	4 26.0 1.2
40	9.26 739	67	9.27 496	70	0.72 504	9.99 243	2	20	5 32.5 1.5
41	9.26 806	67	9.27 566	69	0.72 434	9.99 241	2	19	6 39.0 1.8
42	9.26 873	67	9.27 635	69	0.72 365	9.99 238	3	18	7 45.5 2.1
43	9.26 940	67	9.27 704	69	0.72 296	9.99 236	2	17	8 52.0 2.4
44	9.27 007	66	9.27 773	69	0.72 227	9.99 233	3	16	9 58.5 2.7
45	9.27 073	67	9.27 842	69	0.72 158	9.99 231	2	15	
46	9.27 140	66	9.27 911	69	0.72 089	9.99 229	2	14	3 3 3
47	9.27 206	67	9.27 980	69	0.72 020	9.99 226	3	13	74 73 72
48	9.27 273	66	9.28 049	68	0.71 951	9.99 224	2	12	0 12.3 12.2 12.0
49	9.27 339	66	9.28 117	69	0.71 883	9.99 221	3	11	1 37.0 36.5 36.0
50	9.27 405	66	9.28 186	68	0.71 814	9.99 219	2	10	2 61.7 60.8 60.0
51	9.27 471	66	9.28 254	69	0.71 746	9.99 217	2	9	
52	9.27 537	65	9.28 323	68	0.71 677	9.99 214	3	8	
53	9.27 602	66	9.28 391	68	0.71 609	9.99 212	2	7	3 3 3
54	9.27 668	66	9.28 459	68	0.71 541	9.99 209	3	6	71 70 69
55	9.27 734	65	9.28 527	68	0.71 473	9.99 207	2	5	0 11.8 11.7 11.5
56	9.27 799	65	9.28 595	67	0.71 405	9.99 204	3	4	1 35.5 35.0 34.5
57	9.27 864	66	9.28 662	68	0.71 338	9.99 202	2	3	2 59.2 58.3 57.5
58	9.27 930	65	9.28 730	68	0.71 270	9.99 200	3	2	
59	9.27 995	65	9.28 798	67	0.71 202	9.99 197	2	1	
60	9.28 060		9.28 865		0.71 135	9.99 195		0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.28 060	65	9.28 865	68	0.71 135	9.99 195	3	60	
1	9.28 125	65	9.28 933	67	0.71 067	9.99 192	2	59	
2	9.28 190	64	9.29 000	67	0.71 000	9.99 190	3	58	68 67 66
3	9.28 254	65	9.29 067	67	0.70 933	9.99 187	2	57	1 6.8 6.7 6.6
4	9.28 319	65	9.29 134	67	0.70 866	9.99 185	3	56	2 13.6 13.4 13.2
5	9.28 384	64	9.29 201	67	0.70 799	9.99 182	2	55	3 20.4 20.1 19.8
6	9.28 448	64	9.29 268	67	0.70 732	9.99 180	3	54	4 27.2 26.8 26.4
7	9.28 512	65	9.29 335	67	0.70 665	9.99 177	2	53	5 34.0 33.5 33.0
8	9.28 577	64	9.29 402	66	0.70 598	9.99 175	3	52	6 40.8 40.2 39.6
9	9.28 641	64	9.29 468	66	0.70 532	9.99 172	2	51	7 47.6 46.9 46.2
10	9.28 705	64	9.29 535	66	0.70 465	9.99 170	3	50	8 54.4 53.6 52.8
11	9.28 769	64	9.29 601	67	0.70 399	9.99 167	2	49	9 61.2 60.3 59.4
12	9.28 833	63	9.29 668	66	0.70 332	9.99 165	3	48	
13	9.28 896	64	9.29 734	66	0.70 266	9.99 162	2	47	65 64 63
14	9.28 960	64	9.29 800	66	0.70 200	9.99 160	3	46	1 6.5 6.4 6.3
15	9.29 024	63	9.29 866	66	0.70 134	9.99 157	2	45	2 13.0 12.8 12.6
16	9.29 087	63	9.29 932	66	0.70 068	9.99 155	3	44	3 19.5 19.2 18.9
17	9.29 150	64	9.29 998	66	0.70 002	9.99 152	2	43	4 26.0 25.6 25.2
18	9.29 214	63	9.30 064	66	0.69 936	9.99 150	3	42	5 32.5 32.0 31.5
19	9.29 277	63	9.30 130	65	0.69 870	9.99 147	2	41	6 39.0 38.4 37.8
20	9.29 340	63	9.30 195	66	0.69 805	9.99 145	3	40	7 45.5 44.8 44.1
21	9.29 403	63	9.30 261	65	0.69 739	9.99 142	2	39	8 52.0 51.2 50.4
22	9.29 466	63	9.30 326	65	0.69 674	9.99 140	3	38	9 58.5 57.6 56.7
23	9.29 529	62	9.30 391	66	0.69 609	9.99 137	2	37	
24	9.29 591	63	9.30 457	65	0.69 543	9.99 135	3	36	62 61 60
25	9.29 654	62	9.30 522	65	0.69 478	9.99 132	2	35	1 6.2 6.1 6.0
26	9.29 716	63	9.30 587	65	0.69 413	9.99 130	3	34	2 12.4 12.2 12.0
27	9.29 779	62	9.30 652	65	0.69 348	9.99 127	2	33	3 18.6 18.3 18.0
28	9.29 841	62	9.30 717	65	0.69 283	9.99 124	3	32	4 24.8 24.4 24.0
29	9.29 903	63	9.30 782	64	0.69 218	9.99 122	2	31	5 31.0 30.5 30.0
30	9.29 966	62	9.30 846	65	0.69 154	9.99 119	3	30	6 37.2 36.6 36.0
31	9.30 028	62	9.30 911	64	0.69 089	9.99 117	2	29	7 43.4 42.7 42.0
32	9.30 090	61	9.30 975	65	0.69 025	9.99 114	3	28	8 49.6 48.8 48.0
33	9.30 151	62	9.31 040	64	0.68 960	9.99 112	2	27	9 55.8 54.9 54.0
34	9.30 213	62	9.31 104	64	0.68 896	9.99 109	3	26	
35	9.30 275	61	9.31 168	65	0.68 832	9.99 106	2	25	59 8
36	9.30 336	62	9.31 233	64	0.68 767	9.99 104	3	24	1 5.9 0.3
37	9.30 398	61	9.31 297	64	0.68 703	9.99 101	2	23	2 11.8 0.6
38	9.30 459	62	9.31 361	64	0.68 639	9.99 099	3	22	3 17.7 0.9
39	9.30 521	61	9.31 425	64	0.68 575	9.99 096	2	21	4 23.6 1.2
40	9.30 582	61	9.31 489	63	0.68 511	9.99 093	3	20	5 29.5 1.5
41	9.30 643	61	9.31 552	64	0.68 448	9.99 091	2	19	6 35.4 1.8
42	9.30 704	61	9.31 616	63	0.68 384	9.99 088	3	18	7 41.3 2.1
43	9.30 765	61	9.31 679	64	0.68 321	9.99 086	2	17	8 47.2 2.4
44	9.30 826	61	9.31 743	63	0.68 257	9.99 083	3	16	9 53.1 2.7
45	9.30 887	60	9.31 806	64	0.68 194	9.99 080	2	15	
46	9.30 947	61	9.31 870	63	0.68 130	9.99 078	3	14	3 8 3
47	9.31 008	60	9.31 933	63	0.68 067	9.99 075	2	13	1 5.9 0.3
48	9.31 068	61	9.31 996	63	0.68 004	9.99 072	3	12	2 11.8 0.6
49	9.31 129	60	9.32 059	63	0.67 941	9.99 070	2	11	3 17.7 0.9
50	9.31 189	61	9.32 122	63	0.67 878	9.99 067	3	10	4 23.6 1.2
51	9.31 250	60	9.32 185	63	0.67 815	9.99 064	2	9	5 29.5 1.5
52	9.31 310	60	9.32 248	63	0.67 752	9.99 062	3	8	6 35.4 1.8
53	9.31 370	60	9.32 311	62	0.67 689	9.99 059	2	7	7 41.3 2.1
54	9.31 430	60	9.32 373	63	0.67 627	9.99 056	3	6	8 47.2 2.4
55	9.31 490	59	9.32 436	62	0.67 564	9.99 054	2	5	9 53.1 2.7
56	9.31 549	60	9.32 498	63	0.67 502	9.99 051	3	4	
57	9.31 609	60	9.32 561	62	0.67 439	9.99 048	2	3	3 8 3
58	9.31 669	59	9.32 623	62	0.67 377	9.99 046	3	2	1 5.9 0.3
59	9.31 728	60	9.32 685	62	0.67 315	9.99 043	2	1	2 11.8 0.6
60	9.31 788		9.32 747		0.67 253	9.99 040	3	0	3 17.7 0.9
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	P. P.		
0	9.31 788	59	9.32 747	63	0.67 253	9.99 040	2	60		
1	9.31 847	60	9.32 810	62	0.67 190	9.99 038	3	59		
2	9.31 907	59	9.32 872	61	0.67 128	9.99 035	3	58	63	62 61
3	9.31 966	59	9.32 933	62	0.67 067	9.99 032	3	57	1	6.3 6.2 6.1
4	9.32 025	59	9.32 995	62	0.67 005	9.99 030	3	56	2	12.6 12.4 12.2
5	9.32 084	59	9.33 057	62	0.66 943	9.99 027	3	55	3	18.9 18.6 18.3
6	9.32 143	59	9.33 119	61	0.66 881	9.99 024	3	54	4	25.2 24.8 24.4
7	9.32 202	59	9.33 180	62.	0.66 820	9.99 022	3	53	5	31.5 31.0 30.5
8	9.32 261	59	9.33 242	62.	0.66 758	9.99 019	3	52	6	37.8 37.2 36.6
9	9.32 319	58	9.33 303	61	0.66 697	9.99 016	3	51	7	44.1 43.4 42.7
10	9.32 378	59	9.33 365	61	0.66 635	9.99 013	3	50	8	50.4 49.6 48.8
11	9.32 437	58	9.33 426	61	0.66 574	9.99 011	3	49	9	56.7 55.8 54.9
12	9.32 495	58	9.33 487	61	0.66 513	9.99 008	3	48		
13	9.32 553	58	9.33 548	61	0.66 452	9.99 005	3	47	60	59
14	9.32 612	59	9.33 609	61	0.66 391	9.99 002	3	46	1	6.0 5.9
15	9.32 670	58	9.33 670	61	0.66 330	9.99 000	2	45	2	12.0 11.8
16	9.32 728	58	9.33 731	61	0.66 269	9.98 997	3	44	3	18.0 17.7
17	9.32 786	58	9.33 792	61	0.66 208	9.98 994	3	43	4	24.0 23.6
18	9.32 844	58	9.33 853	60	0.66 147	9.98 991	3	42	5	30.0 29.5
19	9.32 902	58	9.33 913	60	0.66 087	9.98 989	2	41	6	36.0 35.4
20	9.32 960	58	9.33 974	61	0.66 026	9.98 986	3	40	7	42.0 41.3
21	9.33 018	57	9.34 034	61	0.65 966	9.98 983	3	39	8	48.0 47.2
22	9.33 075	58	9.34 095	60	0.65 905	9.98 980	3	38	9	54.0 53.1
23	9.33 133	57	9.34 155	60	0.65 845	9.98 978	2	37		
24	9.33 190	58	9.34 215	61	0.65 785	9.98 975	3	36	58	57
25	9.33 248	57	9.34 276	60	0.65 724	9.98 972	3	35	1	5.8 5.7
26	9.33 305	57	9.34 336	60	0.65 664	9.98 969	3	34	2	11.6 11.4
27	9.33 362	58	9.34 396	60	0.65 604	9.98 967	2	33	3	17.4 17.1
28	9.33 420	57	9.34 456	60	0.65 544	9.98 964	3	32	4	23.2 22.8
29	9.33 477	57	9.34 516	60	0.65 484	9.98 961	3	31	5	29.0 28.5
30	9.33 534	57	9.34 576	60	0.65 424	9.98 958	3	30	6	34.8 34.2
31	9.33 591	56	9.34 635	59	0.65 365	9.98 955	3	29	7	40.6 39.9
32	9.33 647	57	9.34 695	60	0.65 305	9.98 953	2	28	8	46.4 45.6
33	9.33 704	57	9.34 755	60	0.65 245	9.98 950	3	27	9	52.2 51.3
34	9.33 761	57	9.34 814	59	0.65 186	9.98 947	3	26		
35	9.33 818	56	9.34 874	59	0.65 126	9.98 944	3	25	56	55 3
36	9.33 874	57	9.34 933	59	0.65 067	9.98 941	3	24	1	5.6 5.5 0.3
37	9.33 931	56	9.34 992	59	0.65 008	9.98 938	3	23	2	11.2 11.0 0.6
38	9.33 987	56	9.35 051	59	0.64 949	9.98 936	2	22	3	16.8 16.5 0.9
39	9.34 043	57	9.35 111	60	0.64 889	9.98 933	3	21	4	22.4 22.0 1.2
40	9.34 100	56	9.35 170	59	0.64 830	9.98 930	3	20	5	28.0 27.5 1.5
41	9.34 156	56	9.35 229	59	0.64 771	9.98 927	3	19	6	33.6 33.0 1.8
42	9.34 212	56	9.35 288	59	0.64 712	9.98 924	3	18	7	39.2 38.5 2.1
43	9.34 268	56	9.35 347	58	0.64 653	9.98 921	3	17	8	44.8 44.0 2.4
44	9.34 324	56	9.35 405	59	0.64 595	9.98 919	2	16	9	50.4 49.5 2.7
45	9.34 380	56	9.35 464	59	0.64 536	9.98 916	3	15		
46	9.34 436	55	9.35 523	58	0.64 477	9.98 913	3	14	3	3
47	9.34 491	56	9.35 581	59	0.64 419	9.98 910	3	13	0	62 61 60
48	9.34 547	55	9.35 640	58	0.64 360	9.98 907	3	12	1	10.3 10.2 10.0
49	9.34 602	56	9.35 698	59	0.64 302	9.98 904	3	11	2	31.0 30.5 30.0
50	9.34 658	55	9.35 757	58	0.64 243	9.98 901	3	10	3	51.7 50.8 50.0
51	9.34 713	56	9.35 815	58	0.64 185	9.98 898	2	9		
52	9.34 769	55	9.35 873	58	0.64 127	9.98 896	3	8		
53	9.34 824	55	9.35 931	58	0.64 069	9.98 893	3	7	3	3
54	9.34 879	55	9.35 989	58	0.64 011	9.98 890	3	6	0	59 58 57
55	9.34 934	55	9.36 047	58	0.63 953	9.98 887	3	5	1	9.8 9.7 9
56	9.34 989	55	9.36 105	58	0.63 895	9.98 884	3	4	2	29.5 29.0 28
57	9.35 044	55	9.36 163	58	0.63 837	9.98 881	3	3	3	49.2 48.3 47
58	9.35 099	55	9.36 221	58	0.63 779	9.98 878	3	2		
59	9.35 154	55	9.36 279	57	0.63 721	9.98 875	3	1		
60	9.35 209	55	9.36 336	58	0.63 664	9.98 872	3	0		
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.	

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.35 209		9.36 336		0.63 664	9.98 872		60	
1	9.35 263	54	9.36 394	58	0.63 606	9.98 869	3	59	58 57 56
2	9.35 318	55	9.36 452	58	0.63 548	9.98 867	2	58	
3	9.35 373	55	9.36 509	57	0.63 491	9.98 864	3	57	1 5.8 5.7 5.6
4	9.35 427	54	9.36 566	57	0.63 434	9.98 861	3	56	2 11.6 11.4 11.2
5	9.35 481	54	9.36 624	58	0.63 376	9.98 858	3	55	3 17.4 17.1 16.8
6	9.35 536	55	9.36 681	57	0.63 319	9.98 855	3	54	4 23.2 22.8 22.4
7	9.35 590	54	9.36 738	57	0.63 262	9.98 852	3	53	5 29.0 28.5 28.0
8	9.35 644	54	9.36 795	57	0.63 205	9.98 849	3	52	6 34.8 34.2 33.6
9	9.35 698	54	9.36 852	57	0.63 148	9.98 846	3	51	7 40.6 39.9 39.2
10	9.35 752	54	9.36 909	57	0.63 091	9.98 843	3	50	8 46.4 45.6 44.8
11	9.35 806	54	9.36 966	57	0.63 034	9.98 840	3	49	9 52.2 51.3 50.4
12	9.35 860	54	9.37 023	57	0.62 977	9.98 837	3	48	
13	9.35 914	54	9.37 080	57	0.62 920	9.98 834	3	47	55 54 53
14	9.35 968	54	9.37 137	56	0.62 863	9.98 831	3	46	1 5.5 5.4 5.3
15	9.36 022	54	9.37 193	56	0.62 807	9.98 828	3	45	2 11.0 10.8 10.6
16	9.36 075	53	9.37 250	57	0.62 750	9.98 825	3	44	3 16.5 16.2 15.9
17	9.36 129	54	9.37 306	57	0.62 694	9.98 822	3	43	4 22.0 21.6 21.2
18	9.36 182	53	9.37 363	57	0.62 637	9.98 819	3	42	5 27.5 27.0 26.5
19	9.36 236	54	9.37 419	56	0.62 581	9.98 816	3	41	6 33.0 32.4 31.8
20	9.36 289	53	9.37 476	56	0.62 524	9.98 813	3	40	7 38.5 37.8 37.1
21	9.36 342	53	9.37 532	56	0.62 468	9.98 810	3	39	8 44.0 43.2 42.4
22	9.36 395	53	9.37 588	56	0.62 412	9.98 807	3	38	9 49.5 48.6 47.7
23	9.36 449	54	9.37 644	56	0.62 356	9.98 804	3	37	
24	9.36 502	53	9.37 700	56	0.62 300	9.98 801	3	36	52 51
25	9.36 555	53	9.37 756	56	0.62 244	9.98 798	3	35	1 5.2 5.1
26	9.36 608	53	9.37 812	56	0.62 188	9.98 795	3	34	2 10.4 10.2
27	9.36 660	52	9.37 868	56	0.62 132	9.98 792	3	33	3 15.6 15.3
28	9.36 713	53	9.37 924	56	0.62 076	9.98 789	3	32	4 20.8 20.4
29	9.36 766	53	9.37 980	56	0.62 020	9.98 786	3	31	5 26.0 25.5
30	9.36 819	52	9.38 035	55	0.61 963	9.98 783	3	30	6 31.2 30.6
31	9.36 871	53	9.38 091	56	0.61 909	9.98 780	3	29	7 36.4 35.7
32	9.36 924	53	9.38 147	55	0.61 853	9.98 777	3	28	8 41.6 40.8
33	9.36 976	52	9.38 202	55	0.61 798	9.98 774	3	27	9 46.8 45.9
34	9.37 028	52	9.38 257	55	0.61 743	9.98 771	3	26	
35	9.37 081	53	9.38 313	55	0.61 687	9.98 768	3	25	4 3
36	9.37 133	52	9.38 368	55	0.61 632	9.98 765	3	24	1 0.4 0.3
37	9.37 185	52	9.38 423	55	0.61 577	9.98 762	3	23	2 0.8 0.6
38	9.37 237	52	9.38 479	55	0.61 521	9.98 759	3	22	3 1.2 0.9
39	9.37 289	52	9.38 534	55	0.61 466	9.98 756	3	21	4 1.6 1.2
40	9.37 341	52	9.38 589	55	0.61 411	9.98 753	3	20	5 2.0 1.5
41	9.37 393	52	9.38 644	55	0.61 356	9.98 750	3	19	6 2.4 1.8
42	9.37 445	52	9.38 699	55	0.61 301	9.98 746	4	18	7 2.8 2.1
43	9.37 497	52	9.38 754	55	0.61 246	9.98 743	3	17	8 3.2 2.4
44	9.37 549	51	9.38 808	54	0.61 192	9.98 740	3	16	9 3.6 2.7
45	9.37 600	52	9.38 863	55	0.61 137	9.98 737	3	15	
46	9.37 652	52	9.38 918	55	0.61 082	9.98 734	3	14	4 4 3 3
47	9.37 703	51	9.38 972	54	0.61 028	9.98 731	3	13	55 54 53 57
48	9.37 755	51	9.39 027	55	0.60 973	9.98 728	3	12	0 6.9 6.8 9.7 9.5
49	9.37 806	52	9.39 082	54	0.60 918	9.98 725	3	11	1 20.6 20.2 29.0 28.5
50	9.37 858	51	9.39 136	54	0.60 864	9.98 722	3	10	2 34.4 33.8 48.3 47.5
51	9.37 909	51	9.39 190	55	0.60 810	9.98 719	3	9	3 48.1 47.2 — —
52	9.37 960	51	9.39 245	54	0.60 755	9.98 715	4	8	
53	9.38 011	51	9.39 299	54	0.60 701	9.98 712	3	7	
54	9.38 062	51	9.39 353	54	0.60 647	9.98 709	3	6	3 3 3
55	9.38 113	51	9.39 407	54	0.60 593	9.98 706	3	5	56 55 54
56	9.38 164	51	9.39 461	54	0.60 539	9.98 703	3	4	
57	9.38 215	51	9.39 515	54	0.60 485	9.98 700	3	3	0 9.3 9.2 9.0
58	9.38 266	51	9.39 569	54	0.60 431	9.98 697	3	2	1 28.0 27.5 27.0
59	9.38 317	51	9.39 623	54	0.60 377	9.98 694	3	1	2 46.7 45.8 45.0
60	9.38 368	51	9.39 677	54	0.60 323	9.98 690	4	0	3
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.38 368	50	9.39 677	54	0.60 323	9.98 690	3	60	
1	9.38 418	51	9.39 731	54	0.60 269	9.98 687	3	59	54 58
2	9.38 469	50	9.39 785	53	0.60 215	9.98 684	3	58	
3	9.38 519	51	9.39 838	54	0.60 162	9.98 681	3	57	1 5.4 5.3
4	9.38 570	50	9.39 892	53	0.60 108	9.98 678	3	56	2 10.8 10.6
5	9.38 620	50	9.39 945	54	0.60 053	9.98 675	3	55	3 16.2 15.9
6	9.38 670	50	9.39 999	53	0.60 001	9.98 671	4	54	4 21.6 21.2
7	9.38 721	51	9.40 052	53	0.59 948	9.98 668	3	53	5 27.0 26.5
8	9.38 771	50	9.40 106	54	0.59 894	9.98 665	3	52	6 32.4 31.8
9	9.38 821	50	9.40 159	53	0.59 841	9.98 662	3	51	7 37.8 37.1
10	9.38 871	50	9.40 212	53	0.59 788	9.98 659	3	50	8 43.2 42.4
11	9.38 921	50	9.40 266	54	0.59 734	9.98 656	3	49	9 48.6 47.7
12	9.38 971	50	9.40 319	53	0.59 681	9.98 652	4	48	
13	9.39 021	50	9.40 372	53	0.59 628	9.98 649	3	47	52 51 50
14	9.39 071	50	9.40 425	53	0.59 575	9.98 646	3	46	1 5.2 5.1 5.0
15	9.39 121	49	9.40 478	53	0.59 522	9.98 643	3	45	2 10.4 10.2 10.0
16	9.39 170	50	9.40 531	53	0.59 469	9.98 640	3	44	3 15.6 15.3 15.0
17	9.39 220	50	9.40 584	52	0.59 416	9.98 636	4	43	4 20.8 20.4 20.0
18	9.39 270	49	9.40 636	53	0.59 364	9.98 633	3	42	5 26.0 25.5 25.0
19	9.39 319	50	9.40 689	53	0.59 311	9.98 630	3	41	6 31.2 30.6 30.0
20	9.39 369	49	9.40 742	53	0.59 258	9.98 627	3	40	7 36.4 35.7 35.0
21	9.39 418	49	9.40 795	52	0.59 205	9.98 623	4	39	8 41.6 40.8 40.0
22	9.39 467	50	9.40 847	53	0.59 153	9.98 620	3	38	9 46.8 45.9 45.0
23	9.39 517	49	9.40 900	52	0.59 100	9.98 617	3	37	
24	9.39 566	49	9.40 952	53	0.59 048	9.98 614	3	36	49 48 47
25	9.39 615	49	9.41 005	52	0.58 995	9.98 610	4	35	1 4.9 4.8 4.7
26	9.39 664	49	9.41 057	52	0.58 943	9.98 607	3	34	2 9.8 9.6 9.4
27	9.39 713	49	9.41 109	52	0.58 891	9.98 604	3	33	3 14.7 14.4 14.1
28	9.39 762	49	9.41 161	53	0.58 839	9.98 601	3	32	4 19.6 19.2 18.8
29	9.39 811	49	9.41 214	52	0.58 786	9.98 597	4	31	5 24.5 24.0 23.5
30	9.39 860	49	9.41 266	52	0.58 734	9.98 594	3	30	6 29.4 28.8 28.2
31	9.39 909	49	9.41 318	52	0.58 682	9.98 591	3	29	7 34.3 33.6 32.9
32	9.39 958	48	9.41 370	52	0.58 630	9.98 588	3	28	8 39.2 38.4 37.6
33	9.40 006	48	9.41 422	52	0.58 578	9.98 584	4	27	9 44.1 43.2 42.3
34	9.40 055	48	9.41 474	52	0.58 526	9.98 581	3	26	
35	9.40 103	49	9.41 526	52	0.58 474	9.98 578	3	25	4 3
36	9.40 152	48	9.41 578	52	0.58 422	9.98 574	4	24	1 0.4 0.3
37	9.40 200	49	9.41 629	51	0.58 371	9.98 571	3	23	2 0.8 0.6
38	9.40 249	48	9.41 681	52	0.58 319	9.98 568	3	22	3 1.2 0.9
39	9.40 297	49	9.41 733	52	0.58 267	9.98 565	3	21	4 1.6 1.2
40	9.40 346	48	9.41 784	51	0.58 216	9.98 561	4	20	5 2.0 1.5
41	9.40 394	48	9.41 836	52	0.58 164	9.98 558	3	19	6 2.4 1.8
42	9.40 442	48	9.41 887	52	0.58 113	9.98 555	3	18	7 2.8 2.1
43	9.40 490	48	9.41 939	51	0.58 061	9.98 551	4	17	8 3.2 2.4
44	9.40 538	48	9.41 990	51	0.58 010	9.98 548	3	16	9 3.6 2.7
45	9.40 586	48	9.42 041	52	0.57 959	9.98 545	3	15	
46	9.40 634	48	9.42 093	51	0.57 907	9.98 541	4	14	4 4 4 4
47	9.40 682	48	9.42 144	51	0.57 856	9.98 538	3	13	54 53 52 51
48	9.40 730	48	9.42 195	51	0.57 805	9.98 535	3	12	0 6.8 6.6 6.5 6.4
49	9.40 778	47	9.42 246	51	0.57 754	9.98 531	4	11	1 20.2 19.9 19.5 19.1
50	9.40 825	48	9.42 297	51	0.57 703	9.98 528	3	10	2 33.8 33.1 32.5 31.9
51	9.40 873	48	9.42 348	51	0.57 652	9.98 525	3	9	3 47.2 46.4 45.5 44.6
52	9.40 921	47	9.42 399	51	0.57 601	9.98 521	4	8	
53	9.40 968	48	9.42 450	51	0.57 550	9.98 518	3	7	
54	9.41 016	47	9.42 501	51	0.57 499	9.98 515	3	6	3 3 3
55	9.41 063	48	9.42 552	51	0.57 448	9.98 511	4	5	54 53 52
56	9.41 111	47	9.42 603	50	0.57 397	9.98 508	3	4	0 9.0 8.8 8.7 8.6
57	9.41 158	47	9.42 653	51	0.57 347	9.98 505	3	3	1 27.0 26.5 26.0 25.5
58	9.41 205	47	9.42 704	51	0.57 296	9.98 501	4	2	2 45.0 44.2 43.3 42.4
59	9.41 252	48	9.42 755	50	0.57 245	9.98 498	3	1	
60	9.41 300		9.42 805		0.57 195	9.98 494	4	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.41 300		9.42 805	51	0.57 195	9.98 494	3	60	
1	9.41 347	47	9.42 856	50	0.57 144	9.98 491	3	59	51 50 49
2	9.41 394	47	9.42 906	51	0.57 094	9.98 488	4	58	
3	9.41 441	47	9.42 957	50	0.57 043	9.98 484	3	57	1 5.1 5.0 4.9
4	9.41 488	47	9.43 007	50	0.56 993	9.98 481	3	56	2 10.2 10.0 9.8
5	9.41 535	47	9.43 057	51	0.56 943	9.98 477	4	55	3 15.3 15.0 14.7
6	9.41 582	46	9.43 108	50	0.56 892	9.98 474	3	54	4 20.4 20.0 19.6
7	9.41 628	47	9.43 158	50	0.56 842	9.98 471	3	53	5 25.5 25.0 24.5
8	9.41 675	47	9.43 208	50	0.56 792	9.98 467	4	52	6 30.6 30.0 29.4
9	9.41 722	46	9.43 258	50	0.56 742	9.98 464	3	51	7 35.7 35.0 34.3
10	9.41 768	47	9.43 308	50	0.56 692	9.98 460	4	50	8 40.8 40.0 39.2
11	9.41 815	46	9.43 358	50	0.56 642	9.98 457	3	49	9 45.9 45.0 44.1
12	9.41 861	47	9.43 408	50	0.56 592	9.98 453	4	48	
13	9.41 908	46	9.43 458	50	0.56 542	9.98 450	3	47	48 47 46
14	9.41 954	47	9.43 508	50	0.56 492	9.98 447	4	46	1 4.8 4.7 4.6
15	9.42 001	46	9.43 558	49	0.56 442	9.98 443	3	45	2 9.6 9.4 9.2
16	9.42 047	46	9.43 607	50	0.56 393	9.98 440	4	44	3 14.4 14.1 13.8
17	9.42 093	47	9.43 657	50	0.56 343	9.98 436	3	43	4 19.2 18.8 18.4
18	9.42 140	46	9.43 707	49	0.56 293	9.98 433	4	42	5 24.0 23.5 23.0
19	9.42 186	46	9.43 756	50	0.56 244	9.98 429	3	41	6 28.8 28.2 27.6
20	9.42 232	46	9.43 806	49	0.56 194	9.98 426	4	40	7 33.6 32.9 32.2
21	9.42 278	46	9.43 855	50	0.56 145	9.98 422	3	39	8 38.4 37.6 36.8
22	9.42 324	46	9.43 905	49	0.56 095	9.98 419	4	38	9 43.2 42.3 41.4
23	9.42 370	46	9.43 954	50	0.56 046	9.98 415	3	37	
24	9.42 416	45	9.44 004	49	0.55 996	9.98 412	4	36	45 44
25	9.42 461	46	9.44 053	49	0.55 947	9.98 409	3	35	1 4.5 4.4
26	9.42 507	46	9.44 102	49	0.55 898	9.98 405	4	34	2 9.0 8.8
27	9.42 553	46	9.44 151	50	0.55 849	9.98 402	3	33	3 13.5 13.2
28	9.42 599	45	9.44 201	49	0.55 799	9.98 398	4	32	4 18.0 17.6
29	9.42 644	46	9.44 250	49	0.55 750	9.98 395	3	31	5 22.5 22.0
30	9.42 690	45	9.44 299	49	0.55 701	9.98 391	4	30	6 27.0 26.4
31	9.42 735	46	9.44 348	49	0.55 652	9.98 388	3	29	7 31.5 30.8
32	9.42 781	45	9.44 397	49	0.55 603	9.98 384	4	28	8 36.0 35.2
33	9.42 826	46	9.44 446	49	0.55 554	9.98 381	3	27	9 40.5 39.6
34	9.42 872	45	9.44 495	49	0.55 505	9.98 377	4	26	
35	9.42 917	45	9.44 544	48	0.55 456	9.98 373	3	25	4 4.5 4.4
36	9.42 962	46	9.44 592	49	0.55 408	9.98 370	4	24	2 9.0 8.8
37	9.43 008	45	9.44 641	49	0.55 359	9.98 366	3	23	3 13.5 13.2
38	9.43 053	45	9.44 690	48	0.55 310	9.98 363	4	22	4 18.0 17.6
39	9.43 098	45	9.44 738	49	0.55 262	9.98 359	3	21	5 22.5 22.0
40	9.43 143	45	9.44 787	49	0.55 213	9.98 356	4	20	6 27.0 26.4
41	9.43 188	45	9.44 836	48	0.55 164	9.98 352	3	19	7 31.5 30.8
42	9.43 233	45	9.44 884	49	0.55 116	9.98 349	4	18	8 36.0 35.2
43	9.43 278	45	9.44 933	48	0.55 067	9.98 345	3	17	9 40.5 39.6
44	9.43 323	44	9.44 981	48	0.55 019	9.98 342	4	16	
45	9.43 367	45	9.45 029	49	0.54 971	9.98 338	3	15	4 4.5 4.4
46	9.43 412	45	9.45 078	48	0.54 922	9.98 334	4	14	2 9.0 8.8
47	9.43 457	45	9.45 126	48	0.54 874	9.98 331	3	13	3 13.5 13.2
48	9.43 502	44	9.45 174	48	0.54 826	9.98 327	4	12	4 18.0 17.6
49	9.43 546	45	9.45 222	49	0.54 778	9.98 324	3	11	5 22.5 22.0
50	9.43 591	44	9.45 271	48	0.54 729	9.98 320	4	10	6 27.0 26.4
51	9.43 635	45	9.45 319	48	0.54 681	9.98 317	3	9	7 31.5 30.8
52	9.43 680	44	9.45 367	48	0.54 633	9.98 313	4	8	8 36.0 35.2
53	9.43 724	45	9.45 415	48	0.54 585	9.98 309	3	7	9 40.5 39.6
54	9.43 769	44	9.45 463	48	0.54 537	9.98 306	4	6	
55	9.43 813	44	9.45 511	48	0.54 489	9.98 302	3	5	4 4.5 4.4
56	9.43 857	44	9.45 559	47	0.54 441	9.98 299	4	4	2 9.0 8.8
57	9.43 901	45	9.45 606	48	0.54 394	9.98 295	3	3	3 13.5 13.2
58	9.43 946	44	9.45 654	48	0.54 346	9.98 291	4	2	4 18.0 17.6
59	9.43 990	44	9.45 702	48	0.54 298	9.98 288	3	1	5 22.5 22.0
60	9.44 034	44	9.45 750	48	0.54 250	9.98 284	4	0	6 27.0 26.4
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.44 034	44	9.45 750	47	0.54 250	9.98 284	3	60	
1	9.44 078	44	9.45 797	48	0.54 203	9.98 281	3	59	48 47 46
2	9.44 122	44	9.45 845	47	0.54 155	9.98 277	4	58	
3	9.44 166	44	9.45 892	48	0.54 108	9.98 273	4	57	1 4.8 4.7 4.6
4	9.44 210	44	9.45 940	47	0.54 060	9.98 270	3	56	2 9.6 9.4 9.2
5	9.44 253	43	9.45 987	47	0.54 013	9.98 266	4	55	3 14.4 14.1 13.8
6	9.44 297	44	9.46 035	48	0.53 965	9.98 262	4	54	4 19.2 18.8 18.4
7	9.44 341	44	9.46 082	47	0.53 918	9.98 259	3	53	5 24.0 23.5 23.0
8	9.44 385	44	9.46 130	48	0.53 870	9.98 255	4	52	6 28.8 28.2 27.6
9	9.44 428	43	9.46 177	47	0.53 823	9.98 251	4	51	7 33.6 32.9 32.2
10	9.44 472	44	9.46 224	47	0.53 776	9.98 248	3	50	8 38.4 37.6 36.8
11	9.44 516	44	9.46 271	47	0.53 729	9.98 244	4	49	9 43.2 42.3 41.4
12	9.44 559	43	9.46 319	48	0.53 681	9.98 240	4	48	
13	9.44 602	43	9.46 366	47	0.53 634	9.98 237	3	47	45 44 43
14	9.44 646	44	9.46 413	47	0.53 587	9.98 233	4	46	1 4.5 4.4 4.3
15	9.44 689	43	9.46 460	47	0.53 540	9.98 229	4	45	2 9.0 8.8 8.6
16	9.44 733	44	9.46 507	47	0.53 493	9.98 226	3	44	3 13.5 13.2 12.9
17	9.44 776	43	9.46 554	47	0.53 446	9.98 222	4	43	4 18.0 17.6 17.2
18	9.44 819	43	9.46 601	47	0.53 399	9.98 218	4	42	5 22.5 22.0 21.5
19	9.44 862	43	9.46 648	47	0.53 352	9.98 215	3	41	6 27.0 26.4 25.8
20	9.44 905	43	9.46 694	46	0.53 306	9.98 211	4	40	7 31.5 30.8 30.1
21	9.44 948	44	9.46 741	47	0.53 259	9.98 207	4	39	8 36.0 35.2 34.4
22	9.44 992	44	9.46 788	47	0.53 212	9.98 204	3	38	9 40.5 39.6 38.7
23	9.45 035	43	9.46 835	47	0.53 165	9.98 200	4	37	
24	9.45 077	43	9.46 881	47	0.53 119	9.98 196	4	36	42 41
25	9.45 120	43	9.46 928	47	0.53 072	9.98 192	4	35	1 4.2 4.1
26	9.45 163	43	9.46 975	47	0.53 025	9.98 189	3	34	2 8.4 8.2
27	9.45 206	43	9.47 021	46	0.52 979	9.98 185	4	33	3 12.6 12.3
28	9.45 249	43	9.47 068	47	0.52 932	9.98 181	4	32	4 16.8 16.4
29	9.45 292	42	9.47 114	46	0.52 886	9.98 177	4	31	5 21.0 20.5
30	9.45 334	43	9.47 160	47	0.52 840	9.98 174	3	30	6 25.2 24.6
31	9.45 377	42	9.47 207	46	0.52 793	9.98 170	4	29	7 29.4 28.7
32	9.45 419	43	9.47 253	46	0.52 747	9.98 166	4	28	8 33.6 32.8
33	9.45 462	42	9.47 299	47	0.52 701	9.98 162	4	27	9 37.8 36.9
34	9.45 504	43	9.47 346	46	0.52 654	9.98 159	3	26	
35	9.45 547	42	9.47 392	46	0.52 608	9.98 155	4	25	4 3
36	9.45 589	42	9.47 438	46	0.52 562	9.98 151	4	24	1 0.4 0.3
37	9.45 632	43	9.47 484	46	0.52 516	9.98 147	4	23	2 0.8 0.6
38	9.45 674	42	9.47 530	46	0.52 470	9.98 144	3	22	3 1.2 0.9
39	9.45 716	42	9.47 576	46	0.52 424	9.98 140	4	21	4 1.6 1.2
40	9.45 758	43	9.47 622	46	0.52 378	9.98 136	4	20	5 2.0 1.5
41	9.45 801	42	9.47 668	46	0.52 332	9.98 132	4	19	6 2.4 1.8
42	9.45 843	42	9.47 714	46	0.52 286	9.98 129	3	18	7 2.8 2.1
43	9.45 885	42	9.47 760	46	0.52 240	9.98 125	4	17	8 3.2 2.4
44	9.45 927	42	9.47 806	46	0.52 194	9.98 121	4	16	9 3.6 2.7
45	9.45 969	42	9.47 852	45	0.52 148	9.98 117	4	15	
46	9.46 011	42	9.47 897	46	0.52 103	9.98 113	4	14	4 4 4 4
47	9.46 053	42	9.47 943	46	0.52 057	9.98 110	3	13	48 47 46 45
48	9.46 095	41	9.47 989	46	0.52 011	9.98 106	4	12	0 6.0 5.9 5.8 5.6
49	9.46 136	42	9.48 035	45	0.51 965	9.98 102	4	11	1 18.0 17.6 17.2 16.9
50	9.46 178	42	9.48 080	46	0.51 920	9.98 098	4	10	2 30.0 29.4 28.8 28.1
51	9.46 220	42	9.48 126	45	0.51 874	9.98 094	4	9	3 42.0 41.1 40.2 39.4
52	9.46 262	41	9.48 171	46	0.51 829	9.98 090	4	8	
53	9.46 303	42	9.48 217	45	0.51 783	9.98 087	3	7	3 3 3
54	9.46 345	41	9.48 262	45	0.51 738	9.98 083	4	6	48 47 46
55	9.46 386	42	9.48 307	46	0.51 693	9.98 079	4	5	
56	9.46 428	41	9.48 353	45	0.51 647	9.98 075	4	4	0 8.0 7.8 7.7
57	9.46 469	42	9.48 398	45	0.51 602	9.98 071	4	3	1 24.0 23.5 23.0
58	9.46 511	41	9.48 443	46	0.51 557	9.98 067	4	2	2 40.0 39.2 38.3
59	9.46 552	42	9.48 489	45	0.51 511	9.98 063	4	1	
60	9.46 594		9.48 534		0.51 466	9.98 060	3	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.46 594	41	9.48 534	45	0.51 466	9.98 060	4	60	
1	9.46 635	41	9.48 579	45	0.51 421	9.98 056	4	59	
2	9.46 676	41	9.48 624	45	0.51 376	9.98 052	4	58	
3	9.46 717	41	9.48 669	45	0.51 331	9.98 048	4	57	
4	9.46 758	42	9.48 714	45	0.51 286	9.98 044	4	56	
5	9.46 800	42	9.48 759	45	0.51 241	9.98 040	4	55	
6	9.46 841	41	9.48 804	45	0.51 196	9.98 036	4	54	
7	9.46 882	41	9.48 849	45	0.51 151	9.98 032	4	53	
8	9.46 923	41	9.48 894	45	0.51 106	9.98 029	3	52	
9	9.46 964	41	9.48 939	45	0.51 061	9.98 025	4	51	
10	9.47 005	40	9.48 984	45	0.51 016	9.98 021	4	50	
11	9.47 045	41	9.49 029	44	0.50 971	9.98 017	4	49	
12	9.47 086	41	9.49 073	44	0.50 927	9.98 013	4	48	
13	9.47 127	41	9.49 118	45	0.50 882	9.98 009	4	47	
14	9.47 168	41	9.49 163	45	0.50 837	9.98 005	4	46	
15	9.47 209	40	9.49 207	44	0.50 793	9.98 001	4	45	
16	9.47 249	40	9.49 252	45	0.50 748	9.97 997	4	44	
17	9.47 290	41	9.49 296	44	0.50 704	9.97 993	4	43	
18	9.47 330	40	9.49 341	44	0.50 659	9.97 989	4	42	
19	9.47 371	40	9.49 385	44	0.50 615	9.97 986	3	41	
20	9.47 411	41	9.49 430	45	0.50 570	9.97 982	4	40	
21	9.47 452	40	9.49 474	45	0.50 526	9.97 978	4	39	
22	9.47 492	41	9.49 519	44	0.50 481	9.97 974	4	38	
23	9.47 533	40	9.49 563	44	0.50 437	9.97 970	4	37	
24	9.47 573	40	9.49 607	45	0.50 393	9.97 966	4	36	
25	9.47 613	40	9.49 652	44	0.50 348	9.97 962	4	35	
26	9.47 654	41	9.49 696	44	0.50 304	9.97 958	4	34	
27	9.47 694	40	9.49 740	44	0.50 260	9.97 954	4	33	
28	9.47 734	40	9.49 784	44	0.50 216	9.97 950	4	32	
29	9.47 774	40	9.49 828	44	0.50 172	9.97 946	4	31	
30	9.47 814	40	9.49 872	44	0.50 128	9.97 942	4	30	
31	9.47 854	40	9.49 916	44	0.50 084	9.97 938	4	29	
32	9.47 894	40	9.49 960	44	0.50 040	9.97 934	4	28	
33	9.47 934	40	9.50 004	44	0.49 996	9.97 930	4	27	
34	9.47 974	40	9.50 048	44	0.49 952	9.97 926	4	26	
35	9.48 014	40	9.50 092	44	0.49 908	9.97 922	4	25	
36	9.48 054	40	9.50 136	44	0.49 864	9.97 918	4	24	
37	9.48 094	40	9.50 180	44	0.49 820	9.97 914	4	23	
38	9.48 133	39	9.50 223	43	0.49 777	9.97 910	4	22	
39	9.48 173	40	9.50 267	44	0.49 733	9.97 906	4	21	
40	9.48 213	39	9.50 311	44	0.49 689	9.97 902	4	20	
41	9.48 252	40	9.50 355	43	0.49 645	9.97 898	4	19	
42	9.48 292	40	9.50 398	44	0.49 602	9.97 894	4	18	
43	9.48 332	39	9.50 442	43	0.49 558	9.97 890	4	17	
44	9.48 371	40	9.50 485	44	0.49 515	9.97 886	4	16	
45	9.48 411	39	9.50 529	43	0.49 471	9.97 882	4	15	
46	9.48 450	40	9.50 572	44	0.49 428	9.97 878	4	14	
47	9.48 490	39	9.50 616	43	0.49 384	9.97 874	4	13	
48	9.48 529	39	9.50 659	43	0.49 341	9.97 870	4	12	
49	9.48 568	39	9.50 703	43	0.49 297	9.97 866	4	11	
50	9.48 607	40	9.50 746	43	0.49 254	9.97 861	5	10	
51	9.48 647	39	9.50 789	44	0.49 211	9.97 857	4	9	
52	9.48 686	39	9.50 833	43	0.49 167	9.97 853	4	8	
53	9.48 725	39	9.50 876	43	0.49 124	9.97 849	4	7	
54	9.48 764	39	9.50 919	43	0.49 081	9.97 845	4	6	
55	9.48 803	39	9.50 962	43	0.49 038	9.97 841	4	5	
56	9.48 842	39	9.51 005	43	0.48 995	9.97 837	4	4	
57	9.48 881	39	9.51 048	44	0.48 952	9.97 833	4	3	
58	9.48 920	39	9.51 092	44	0.48 908	9.97 829	4	2	
59	9.48 959	39	9.51 135	43	0.48 865	9.97 825	4	1	
60	9.48 998	39	9.51 178	43	0.48 822	9.97 821	4	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.48 998	39	9.51 178	43	0.48 822	9.97 821	4	60	
1	9.49 037	39	9.51 221	43	0.48 779	9.97 817	4	59	
2	9.49 076	39	9.51 264	42	0.48 736	9.97 812	5	58	
3	9.49 115	38	9.51 306	43	0.48 694	9.97 808	4	57	43 43 41
4	9.49 153	39	9.51 349	43	0.48 651	9.97 804	4	56	1 4.3 4.2 4.1
5	9.49 192	39	9.51 392	43	0.48 608	9.97 800	4	55	2 8.6 8.4 8.2
6	9.49 231	38	9.51 435	43	0.48 565	9.97 796	4	54	3 12.9 12.6 12.3
7	9.49 269	39	9.51 478	43	0.48 522	9.97 792	4	53	4 17.2 16.8 16.4
8	9.49 308	39	9.51 520	42	0.48 480	9.97 788	4	52	5 21.5 21.0 20.5
9	9.49 347	38	9.51 563	43	0.48 437	9.97 784	4	51	6 25.8 25.2 24.6
10	9.49 385	39	9.51 606	42	0.48 394	9.97 779	5	50	7 30.1 29.4 28.7
11	9.49 424	38	9.51 648	43	0.48 352	9.97 775	4	49	8 34.4 33.6 32.8
12	9.49 462	38	9.51 691	43	0.48 309	9.97 771	4	48	9 38.7 37.8 36.9
13	9.49 500	39	9.51 734	42	0.48 266	9.97 767	4	47	
14	9.49 539	38	9.51 776	43	0.48 224	9.97 763	4	46	
15	9.49 577	38	9.51 819	42	0.48 181	9.97 759	4	45	89 38 37
16	9.49 615	39	9.51 861	42	0.48 139	9.97 754	5	44	1 3.9 3.8 3.7
17	9.49 654	38	9.51 903	43	0.48 097	9.97 750	4	43	2 7.8 7.6 7.4
18	9.49 692	38	9.51 946	42	0.48 054	9.97 746	4	42	3 11.7 11.4 11.1
19	9.49 730	38	9.51 988	43	0.48 012	9.97 742	4	41	4 15.6 15.2 14.8
20	9.49 768	38	9.52 031	42	0.47 969	9.97 738	4	40	5 19.5 19.0 18.5
21	9.49 806	38	9.52 073	42	0.47 927	9.97 734	5	39	6 23.4 22.8 22.2
22	9.49 844	38	9.52 115	42	0.47 885	9.97 729	4	38	7 27.3 26.6 25.9
23	9.49 882	38	9.52 157	43	0.47 843	9.97 725	4	37	8 31.2 30.4 29.6
24	9.49 920	38	9.52 200	42	0.47 800	9.97 721	4	36	9 35.1 34.2 33.3
25	9.49 958	38	9.52 242	42	0.47 758	9.97 717	4	35	
26	9.49 996	38	9.52 284	42	0.47 716	9.97 713	5	34	
27	9.50 034	38	9.52 326	42	0.47 674	9.97 708	4	33	36 5 4
28	9.50 072	38	9.52 368	42	0.47 632	9.97 704	4	32	1 3.6 0.5 0.4
29	9.50 110	38	9.52 410	42	0.47 590	9.97 700	4	31	2 7.2 1.0 0.8
30	9.50 148	37	9.52 452	42	0.47 548	9.97 696	5	30	3 10.8 1.5 1.2
31	9.50 185	38	9.52 494	42	0.47 506	9.97 691	4	29	4 14.4 2.0 1.6
32	9.50 223	38	9.52 536	42	0.47 464	9.97 687	4	28	5 18.0 2.5 2.0
33	9.50 261	37	9.52 578	42	0.47 422	9.97 683	4	27	6 21.6 3.0 2.4
34	9.50 298	38	9.52 620	41	0.47 380	9.97 679	5	26	7 25.2 3.5 2.8
35	9.50 336	38	9.52 661	42	0.47 339	9.97 674	5	25	8 28.8 4.0 3.2
36	9.50 374	37	9.52 703	42	0.47 297	9.97 670	4	24	9 32.4 4.5 3.6
37	9.50 411	38	9.52 745	42	0.47 255	9.97 666	4	23	
38	9.50 449	37	9.52 787	42	0.47 213	9.97 662	5	22	
39	9.50 486	37	9.52 829	41	0.47 171	9.97 657	4	21	
40	9.50 523	38	9.52 870	42	0.47 130	9.97 653	4	20	
41	9.50 561	37	9.52 912	41	0.47 088	9.97 649	4	19	5 5 5
42	9.50 598	37	9.52 953	42	0.47 047	9.97 645	5	18	43 42 41
43	9.50 635	38	9.52 995	42	0.47 005	9.97 640	4	17	1 4.3 4.2 4.1
44	9.50 673	37	9.53 037	41	0.46 963	9.97 636	4	16	2 12.9 12.6 12.3
45	9.50 710	37	9.53 078	42	0.46 922	9.97 632	4	15	3 21.5 21.0 20.5
46	9.50 747	37	9.53 120	41	0.46 880	9.97 628	5	14	4 30.1 29.4 28.7
47	9.50 784	37	9.53 161	41	0.46 839	9.97 623	4	13	5 38.7 37.8 36.9
48	9.50 821	37	9.53 202	42	0.46 798	9.97 619	4	12	
49	9.50 858	38	9.53 244	41	0.46 756	9.97 615	5	11	
50	9.50 896	37	9.53 285	42	0.46 715	9.97 610	4	10	
51	9.50 933	37	9.53 327	41	0.46 673	9.97 606	4	9	4 4
52	9.50 970	37	9.53 368	41	0.46 632	9.97 602	5	8	43 42
53	9.51 007	36	9.53 409	42	0.46 591	9.97 597	4	7	
54	9.51 043	37	9.53 450	41	0.46 550	9.97 593	4	6	0 5.4 5.2 5.1
55	9.51 080	37	9.53 492	41	0.46 508	9.97 589	5	5	1 16.1 15.8 15.5
56	9.51 117	37	9.53 533	41	0.46 467	9.97 584	4	4	2 26.9 26.2 25.5
57	9.51 154	37	9.53 574	41	0.46 426	9.97 580	4	3	3 37.6 36.8 36.0
58	9.51 191	36	9.53 615	41	0.46 385	9.97 576	5	2	
59	9.51 227	37	9.53 656	41	0.46 344	9.97 571	4	1	
60	9.51 264	37	9.53 697		0.46 303	9.97 567		0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.51 264	37	9.53 697	41	0.46 303	9.97 567	4	60	
1	9.51 301	37	9.53 738	41	0.46 262	9.97 563	5	59	
2	9.51 338	36	9.53 779	41	0.46 221	9.97 558	5	58	
3	9.51 374	36	9.53 820	41	0.46 180	9.97 554	4	57	41 40 39
4	9.51 411	37	9.53 861	41	0.46 139	9.97 550	4	56	1 4.1 4.0 3.9
5	9.51 447	36	9.53 902	41	0.46 098	9.97 545	5	55	2 8.2 8.0 7.8
6	9.51 484	37	9.53 943	41	0.46 057	9.97 541	4	54	3 12.3 12.0 11.7
7	9.51 520	36	9.53 984	41	0.46 016	9.97 536	5	53	4 16.4 16.0 15.6
8	9.51 557	37	9.54 025	40	0.45 975	9.97 532	4	52	5 20.5 20.0 19.5
9	9.51 593	36	9.54 065	41	0.45 935	9.97 528	4	51	6 24.6 24.0 23.4
10	9.51 629	37	9.54 106	41	0.45 894	9.97 523	5	50	7 28.7 28.0 27.3
11	9.51 666	36	9.54 147	40	0.45 853	9.97 519	4	49	8 32.8 32.0 31.2
12	9.51 702	36	9.54 187	40	0.45 813	9.97 515	4	48	9 36.9 36.0 35.1
13	9.51 738	36	9.54 228	41	0.45 772	9.97 510	5	47	
14	9.51 774	37	9.54 269	40	0.45 731	9.97 506	4	46	
15	9.51 811	36	9.54 309	40	0.45 691	9.97 501	5	45	37 36 35
16	9.51 847	36	9.54 350	40	0.45 650	9.97 497	4	44	1 3.7 3.6 3.5
17	9.51 883	36	9.54 390	40	0.45 610	9.97 492	5	43	2 7.4 7.2 7.0
18	9.51 919	36	9.54 431	41	0.45 569	9.97 488	4	42	3 11.1 10.8 10.5
19	9.51 955	36	9.54 471	41	0.45 529	9.97 484	4	41	4 14.8 14.4 14.0
20	9.51 991	36	9.54 512	40	0.45 488	9.97 479	5	40	5 18.5 18.0 17.5
21	9.52 027	36	9.54 552	41	0.45 448	9.97 475	6	39	6 22.2 21.6 21.0
22	9.52 063	36	9.54 593	40	0.45 407	9.97 470	5	38	7 25.9 25.2 24.5
23	9.52 099	36	9.54 633	40	0.45 367	9.97 466	4	37	8 29.6 28.8 28.0
24	9.52 135	36	9.54 673	41	0.45 327	9.97 461	5	36	9 33.3 32.4 31.5
25	9.52 171	36	9.54 714	40	0.45 286	9.97 457	4	35	
26	9.52 207	35	9.54 754	40	0.45 246	9.97 453	4	34	
27	9.52 242	36	9.54 794	41	0.45 206	9.97 448	5	33	34 5 4
28	9.52 278	36	9.54 835	40	0.45 165	9.97 444	4	32	1 3.4 0.5 0.4
29	9.52 314	36	9.54 875	40	0.45 125	9.97 439	5	31	2 6.8 1.0 0.8
30	9.52 350	35	9.54 915	40	0.45 085	9.97 435	4	30	3 10.2 1.5 1.2
31	9.52 385	36	9.54 955	40	0.45 045	9.97 430	5	29	4 13.6 2.0 1.6
32	9.52 421	35	9.54 995	40	0.45 005	9.97 426	4	28	5 17.0 2.5 2.0
33	9.52 456	36	9.55 035	40	0.44 965	9.97 421	5	27	6 20.4 3.0 2.4
34	9.52 492	35	9.55 075	40	0.44 925	9.97 417	4	26	7 23.8 3.5 2.8
35	9.52 527	36	9.55 115	40	0.44 885	9.97 412	5	25	8 27.2 4.0 3.2
36	9.52 563	35	9.55 155	40	0.44 845	9.97 408	4	24	9 30.6 4.5 3.6
37	9.52 598	36	9.55 195	40	0.44 805	9.97 403	5	23	
38	9.52 634	35	9.55 235	40	0.44 765	9.97 399	4	22	
39	9.52 669	36	9.55 275	40	0.44 725	9.97 394	5	21	
40	9.52 705	35	9.55 315	40	0.44 685	9.97 390	4	20	
41	9.52 740	35	9.55 355	40	0.44 645	9.97 385	5	19	5 5 5
42	9.52 775	36	9.55 395	39	0.44 605	9.97 381	4	18	41 40 39
43	9.52 811	35	9.55 434	40	0.44 566	9.97 376	5	17	
44	9.52 846	35	9.55 474	40	0.44 526	9.97 372	4	16	0 4.1 4.0 3.9
45	9.52 881	35	9.55 514	40	0.44 486	9.97 367	5	15	1 12.3 12.0 11.7
46	9.52 916	35	9.55 554	39	0.44 446	9.97 363	4	14	2 20.5 20.0 19.5
47	9.52 951	35	9.55 593	40	0.44 407	9.97 358	5	13	3 28.7 28.0 27.3
48	9.52 986	35	9.55 633	40	0.44 367	9.97 353	4	12	4 36.9 36.0 35.1
49	9.53 021	35	9.55 673	39	0.44 327	9.97 349	5	11	
50	9.53 056	36	9.55 712	40	0.44 288	9.97 344	4	10	
51	9.53 092	34	9.55 752	39	0.44 248	9.97 340	5	9	4 4 4
52	9.53 126	35	9.55 791	40	0.44 209	9.97 335	4	8	41 40 39
53	9.53 161	35	9.55 831	39	0.44 169	9.97 331	5	7	
54	9.53 196	35	9.55 870	40	0.44 130	9.97 326	4	6	0 5.1 5.0 4.9
55	9.53 231	35	9.55 910	39	0.44 090	9.97 322	5	5	1 15.4 15.0 14.6
56	9.53 266	35	9.55 949	40	0.44 051	9.97 317	4	4	2 25.6 25.0 24.4
57	9.53 301	35	9.55 989	39	0.44 011	9.97 312	5	3	3 35.9 35.0 34.1
58	9.53 336	34	9.56 028	39	0.43 972	9.97 308	4	2	
59	9.53 370	35	9.56 067	40	0.43 933	9.97 303	5	1	
60	9.53 405	35	9.56 107	40	0.43 893	9.97 299	4	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.53 405	35	9.56 107	39	0.43 893	9.97 299	5	60	
1	9.53 440	35	9.56 146	39	0.43 854	9.97 294	5	59	
2	9.53 475	34	9.56 185	39	0.43 815	9.97 289	5	58	40 39 38
3	9.53 509	34	9.56 224	39	0.43 776	9.97 285	4	57	
4	9.53 544	35	9.56 264	40	0.43 736	9.97 280	5	56	I 4.0 3.9 3.8
5	9.53 578	34	9.56 303	39	0.43 697	9.97 276	4	55	2 8.0 7.8 7.6
6	9.53 613	35	9.56 342	39	0.43 658	9.97 271	5	54	3 12.0 11.7 11.4
7	9.53 647	34	9.56 381	39	0.43 619	9.97 266	5	53	4 16.0 15.6 15.2
8	9.53 682	35	9.56 420	39	0.43 580	9.97 262	4	52	5 20.0 19.5 19.0
9	9.53 716	34	9.56 459	39	0.43 541	9.97 257	5	51	6 24.0 23.4 22.8
10	9.53 751	35	9.56 498	39	0.43 502	9.97 252	5	50	7 28.0 27.3 26.6
11	9.53 785	34	9.56 537	39	0.43 463	9.97 248	4	49	8 32.0 31.2 30.4
12	9.53 819	34	9.56 576	39	0.43 424	9.97 243	5	48	9 36.0 35.1 34.2
13	9.53 854	35	9.56 615	39	0.43 385	9.97 238	5	47	
14	9.53 888	34	9.56 654	39	0.43 346	9.97 234	4	46	37 35 34
15	9.53 922	34	9.56 693	39	0.43 307	9.97 229	5	45	
16	9.53 957	35	9.56 732	39	0.43 268	9.97 224	5	44	I 3.7 3.5 3.4
17	9.53 991	34	9.56 771	39	0.43 229	9.97 220	4	43	2 7.4 7.0 6.8
18	9.54 025	34	9.56 810	39	0.43 190	9.97 215	5	42	3 11.1 10.5 10.2
19	9.54 059	34	9.56 849	38	0.43 151	9.97 210	5	41	4 14.8 14.0 13.6
20	9.54 093	34	9.56 887	39	0.43 113	9.97 206	4	40	5 18.5 17.5 17.0
21	9.54 127	34	9.56 926	39	0.43 074	9.97 201	5	39	6 22.2 21.0 20.4
22	9.54 161	34	9.56 965	39	0.43 035	9.97 196	5	38	7 25.9 24.5 23.8
23	9.54 195	34	9.57 004	38	0.42 996	9.97 192	4	37	8 29.6 28.0 27.2
24	9.54 229	34	9.57 042	39	0.42 958	9.97 187	5	36	9 33.3 31.5 30.6
25	9.54 263	34	9.57 081	39	0.42 919	9.97 182	5	35	
26	9.54 297	34	9.57 120	38	0.42 880	9.97 178	4	34	33 5 4
27	9.54 331	34	9.57 158	39	0.42 842	9.97 173	5	33	
28	9.54 365	34	9.57 197	38	0.42 803	9.97 168	5	32	I 3.3 0.5 0.4
29	9.54 399	34	9.57 235	39	0.42 765	9.97 163	5	31	2 6.6 1.0 0.8
30	9.54 433	34	9.57 274	38	0.42 726	9.97 159	4	30	3 9.9 1.5 1.2
31	9.54 466	33	9.57 312	39	0.42 688	9.97 154	5	29	4 13.2 2.0 1.6
32	9.54 500	34	9.57 351	38	0.42 649	9.97 149	5	28	5 16.5 2.5 2.0
33	9.54 534	34	9.57 389	39	0.42 611	9.97 145	4	27	6 19.8 3.0 2.4
34	9.54 567	34	9.57 428	38	0.42 572	9.97 140	5	26	7 23.1 3.5 2.8
35	9.54 601	34	9.57 466	38	0.42 534	9.97 135	5	25	8 26.4 4.0 3.2
36	9.54 635	34	9.57 504	39	0.42 496	9.97 130	5	24	9 29.7 4.5 3.6
37	9.54 668	33	9.57 543	39	0.42 457	9.97 126	4	23	
38	9.54 702	34	9.57 581	38	0.42 419	9.97 121	5	22	
39	9.54 735	33	9.57 619	39	0.42 381	9.97 116	5	21	
40	9.54 769	34	9.57 658	38	0.42 342	9.97 111	5	20	
41	9.54 802	33	9.57 696	38	0.42 304	9.97 107	4	19	5 5 5
42	9.54 836	34	9.57 734	38	0.42 266	9.97 102	5	18	40 39 38
43	9.54 869	33	9.57 772	38	0.42 228	9.97 097	5	17	
44	9.54 903	34	9.57 810	38	0.42 190	9.97 092	5	16	I 4.0 3.9 3.8
45	9.54 936	33	9.57 849	39	0.42 151	9.97 087	5	15	2 12.0 11.7 11.4
46	9.54 969	33	9.57 887	38	0.42 113	9.97 083	4	14	3 20.0 19.5 19.0
47	9.55 003	34	9.57 925	38	0.42 075	9.97 078	5	13	4 28.0 27.3 26.6
48	9.55 036	33	9.57 963	38	0.42 037	9.97 073	5	12	5 36.0 35.1 34.2
49	9.55 069	33	9.58 001	38	0.41 999	9.97 068	5	11	
50	9.55 102	33	9.58 039	38	0.41 961	9.97 063	5	10	
51	9.55 136	34	9.58 077	38	0.41 923	9.97 059	4	9	5 4 4
52	9.55 169	33	9.58 115	38	0.41 885	9.97 054	5	8	37 39 38
53	9.55 202	33	9.58 153	38	0.41 847	9.97 049	5	7	
54	9.55 235	33	9.58 191	38	0.41 809	9.97 044	5	6	I 3.7 4.9 4.8
55	9.55 268	33	9.58 229	38	0.41 771	9.97 039	5	5	2 11.1 14.6 14.2
56	9.55 301	33	9.58 267	38	0.41 733	9.97 035	4	4	3 18.5 24.4 23.8
57	9.55 334	33	9.58 304	37	0.41 696	9.97 030	5	3	4 25.9 34.1 33.2
58	9.55 367	33	9.58 342	38	0.41 658	9.97 025	5	2	5 33.3 — —
59	9.55 400	33	9.58 380	38	0.41 620	9.97 020	5	1	
60	9.55 433	33	9.58 418	38	0.41 582	9.97 015	5	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.55 433	33	9.58 418	37	0.41 582	9.97 015	5	60	
1	9.55 466	33	9.58 455	38	0.41 543	9.97 010	5	59	
2	9.55 499	33	9.58 493	38	0.41 507	9.97 005	5	58	
3	9.55 532	33	9.58 531	38	0.41 469	9.97 001	5	57	38 37 36
4	9.55 564	33	9.58 569	37	0.41 431	9.96 996	5	56	1 3.8 3.7 3.6
5	9.55 597	33	9.58 606	38	0.41 394	9.96 991	5	55	2 7.6 7.4 7.2
6	9.55 630	33	9.58 644	37	0.41 356	9.96 986	5	54	3 11.4 11.1 10.8
7	9.55 663	33	9.58 681	38	0.41 319	9.96 981	5	53	4 15.2 14.8 14.4
8	9.55 695	32	9.58 719	38	0.41 281	9.96 976	5	52	5 19.0 18.5 18.0
9	9.55 728	33	9.58 757	37	0.41 243	9.96 971	5	51	6 22.8 22.2 21.6
10	9.55 761	32	9.58 794	38	0.41 206	9.96 966	5	50	7 26.6 25.9 25.2
11	9.55 793	33	9.58 832	37	0.41 168	9.96 962	4	49	8 30.4 29.6 28.8
12	9.55 826	33	9.58 869	38	0.41 131	9.96 957	5	48	9 34.2 33.3 32.4
13	9.55 858	33	9.58 907	37	0.41 093	9.96 952	5	47	
14	9.55 891	32	9.58 944	37	0.41 056	9.96 947	5	46	33 32 31
15	9.55 923	33	9.58 981	38	0.41 019	9.96 942	5	45	
16	9.55 956	32	9.59 019	37	0.40 981	9.96 937	5	44	1 3.3 3.2 3.1
17	9.55 988	33	9.59 056	38	0.40 944	9.96 932	5	43	2 6.6 6.4 6.2
18	9.56 021	32	9.59 094	37	0.40 906	9.96 927	5	42	3 9.9 9.6 9.3
19	9.56 053	32	9.59 131	37	0.40 869	9.96 922	5	41	4 13.2 12.8 12.4
20	9.56 085	33	9.59 168	37	0.40 832	9.96 917	5	40	5 16.5 16.0 15.5
21	9.56 118	32	9.59 205	38	0.40 795	9.96 912	5	39	6 19.8 19.2 18.6
22	9.56 150	32	9.59 243	37	0.40 757	9.96 907	5	38	7 23.1 22.4 21.7
23	9.56 182	33	9.59 280	37	0.40 720	9.96 903	4	37	8 26.4 25.6 24.8
24	9.56 215	33	9.59 317	37	0.40 683	9.96 898	5	36	9 29.7 28.8 27.9
25	9.56 247	32	9.59 354	37	0.40 646	9.96 893	5	35	
26	9.56 279	32	9.59 391	38	0.40 609	9.96 888	5	34	6 5 4
27	9.56 311	32	9.59 429	37	0.40 571	9.96 883	5	33	1 0.6 0.5 0.4
28	9.56 343	32	9.59 466	37	0.40 534	9.96 878	5	32	2 1.2 1.0 0.8
29	9.56 375	33	9.59 503	37	0.40 497	9.96 873	5	31	3 1.8 1.5 1.2
30	9.56 408	32	9.59 540	37	0.40 460	9.96 868	5	30	4 2.4 2.0 1.6
31	9.56 440	32	9.59 577	37	0.40 423	9.96 863	5	29	5 3.0 2.5 2.0
32	9.56 472	32	9.59 614	37	0.40 386	9.96 858	5	28	6 3.6 3.0 2.4
33	9.56 504	32	9.59 651	37	0.40 349	9.96 853	5	27	7 4.2 3.5 2.8
34	9.56 536	32	9.59 688	37	0.40 312	9.96 848	5	26	8 4.8 4.0 3.2
35	9.56 568	31	9.59 725	37	0.40 275	9.96 843	5	25	9 5.4 4.5 3.6
36	9.56 599	32	9.59 762	37	0.40 238	9.96 838	5	24	
37	9.56 631	32	9.59 799	36	0.40 201	9.96 833	5	23	
38	9.56 663	32	9.59 835	37	0.40 165	9.96 828	5	22	
39	9.56 695	32	9.59 872	37	0.40 128	9.96 823	5	21	
40	9.56 727	32	9.59 909	37	0.40 091	9.96 818	5	20	6 5 5
41	9.56 759	31	9.59 946	37	0.40 054	9.96 813	5	19	37 38 37
42	9.56 790	32	9.59 983	36	0.40 017	9.96 808	5	18	
43	9.56 822	32	9.60 019	37	0.39 981	9.96 803	5	17	0 3.1 3.8 3.7
44	9.56 854	32	9.60 056	37	0.39 944	9.96 798	5	16	1 9.2 11.4 11.1
45	9.56 886	31	9.60 093	37	0.39 907	9.96 793	5	15	2 15.4 19.0 18.5
46	9.56 917	32	9.60 130	36	0.39 870	9.96 788	5	14	3 21.6 26.6 25.9
47	9.56 949	31	9.60 166	37	0.39 834	9.96 783	5	13	4 27.8 34.2 33.3
48	9.56 980	32	9.60 203	37	0.39 797	9.96 778	5	12	5 33.9 — —
49	9.57 012	32	9.60 240	36	0.39 760	9.96 772	6	11	
50	9.57 044	31	9.60 276	37	0.39 724	9.96 767	5	10	5 4 4
51	9.57 075	32	9.60 313	36	0.39 687	9.96 762	5	9	38 38 37
52	9.57 107	31	9.60 349	37	0.39 651	9.96 757	5	8	
53	9.57 138	31	9.60 386	36	0.39 614	9.96 752	5	7	0 3.6 4.8 4.6
54	9.57 169	32	9.60 422	37	0.39 578	9.96 747	5	6	1 10.8 14.2 13.9
55	9.57 201	31	9.60 459	36	0.39 541	9.96 742	5	5	2 18.0 23.8 23.1
56	9.57 232	32	9.60 495	37	0.39 505	9.96 737	5	4	3 25.2 33.2 32.4
57	9.57 264	31	9.60 532	36	0.39 468	9.96 732	5	3	4 32.4 — —
58	9.57 295	31	9.60 568	37	0.39 432	9.96 727	5	2	
59	9.57 326	32	9.60 605	36	0.39 395	9.96 722	5	1	
60	9.57 358	32	9.60 641	36	0.39 359	9.96 717	5	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.57 358	31	9.60 641	36	0.39 359	9.96 717	6	60	
1	9.57 389	31	9.60 677	37	0.39 323	9.96 711	5	59	
2	9.57 420	31	9.60 714	36	0.39 286	9.96 706	5	58	37 36 35
3	9.57 451	31	9.60 750	36	0.39 250	9.96 701	5	57	1 3.7 3.6 3.5
4	9.57 482	32	9.60 786	37	0.39 214	9.96 696	5	56	2 7.4 7.2 7.0
5	9.57 514	31	9.60 823	36	0.39 177	9.96 691	5	55	3 11.1 10.8 10.5
6	9.57 545	31	9.60 859	36	0.39 141	9.96 686	5	54	4 14.8 14.4 14.0
7	9.57 576	31	9.60 895	36	0.39 105	9.96 681	5	53	5 18.5 18.0 17.5
8	9.57 607	31	9.60 931	36	0.39 069	9.96 676	5	52	6 22.2 21.6 21.0
9	9.57 638	31	9.60 967	37	0.39 033	9.96 670	6	51	7 25.9 25.2 24.5
10	9.57 669	31	9.61 004	36	0.38 996	9.96 665	5	50	8 29.6 28.8 28.0
11	9.57 700	31	9.61 040	36	0.38 960	9.96 660	5	49	9 33.3 32.4 31.5
12	9.57 731	31	9.61 076	36	0.38 924	9.96 655	5	48	
13	9.57 762	31	9.61 112	36	0.38 888	9.96 650	5	47	
14	9.57 793	31	9.61 148	36	0.38 852	9.96 645	5	46	32 31 30
15	9.57 824	31	9.61 184	36	0.38 816	9.96 640	5	45	1 3.2 3.1 3.0
16	9.57 855	30	9.61 220	36	0.38 780	9.96 634	6	44	2 6.4 6.2 6.0
17	9.57 885	31	9.61 256	36	0.38 744	9.96 629	5	43	3 9.6 9.3 9.0
18	9.57 916	31	9.61 292	36	0.38 708	9.96 624	5	42	4 12.8 12.4 12.0
19	9.57 947	31	9.61 328	36	0.38 672	9.96 619	5	41	5 16.0 15.5 15.0
20	9.57 978	30	9.61 364	36	0.38 636	9.96 614	5	40	6 19.2 18.6 18.0
21	9.58 008	31	9.61 400	36	0.38 600	9.96 608	6	39	7 22.4 21.7 21.0
22	9.58 039	31	9.61 436	36	0.38 564	9.96 603	5	38	8 25.6 24.8 24.0
23	9.58 070	31	9.61 472	36	0.38 528	9.96 598	5	37	9 28.8 27.9 27.0
24	9.58 101	30	9.61 508	36	0.38 492	9.96 593	5	36	
25	9.58 131	31	9.61 544	35	0.38 456	9.96 588	5	35	
26	9.58 162	30	9.61 579	36	0.38 421	9.96 582	6	34	29 6 5
27	9.58 192	31	9.61 615	36	0.38 385	9.96 577	5	33	1 2.9 0.6 0.5
28	9.58 223	31	9.61 651	36	0.38 349	9.96 572	5	32	2 5.8 1.2 1.0
29	9.58 253	31	9.61 687	35	0.38 313	9.96 567	5	31	3 8.7 1.8 1.5
30	9.58 284	30	9.61 722	36	0.38 278	9.96 562	5	30	4 11.6 2.4 2.0
31	9.58 314	31	9.61 758	36	0.38 242	9.96 556	6	29	5 14.5 3.0 2.5
32	9.58 345	30	9.61 794	36	0.38 206	9.96 551	5	28	6 17.4 3.6 3.0
33	9.58 375	31	9.61 830	35	0.38 170	9.96 546	5	27	7 20.3 4.2 3.5
34	9.58 406	30	9.61 865	36	0.38 135	9.96 541	5	26	8 23.2 4.8 4.0
35	9.58 436	31	9.61 901	35	0.38 099	9.96 535	6	25	9 26.1 5.4 4.5
36	9.58 467	30	9.61 936	36	0.38 064	9.96 530	5	24	
37	9.58 497	30	9.61 972	36	0.38 028	9.96 525	5	23	
38	9.58 527	30	9.62 008	35	0.37 992	9.96 520	5	22	
39	9.58 557	31	9.62 043	36	0.37 957	9.96 514	6	21	
40	9.58 588	30	9.62 079	35	0.37 921	9.96 509	5	20	6 6
41	9.58 618	30	9.62 114	36	0.37 886	9.96 504	5	19	36 35
42	9.58 648	30	9.62 150	36	0.37 850	9.96 498	6	18	0 3.0 2.9
43	9.58 678	31	9.62 185	35	0.37 815	9.96 493	5	17	1 9.0 8.8
44	9.58 709	30	9.62 221	36	0.37 779	9.96 488	5	16	2 15.0 14.6
45	9.58 739	30	9.62 256	35	0.37 744	9.96 483	5	15	3 21.0 20.4
46	9.58 769	30	9.62 292	36	0.37 708	9.96 477	6	14	4 27.0 26.2
47	9.58 799	30	9.62 327	35	0.37 673	9.96 472	5	13	5 33.0 32.1
48	9.58 829	30	9.62 362	35	0.37 638	9.96 467	5	12	
49	9.58 859	30	9.62 398	36	0.37 602	9.96 461	6	11	
50	9.58 889	30	9.62 433	35	0.37 567	9.96 456	5	10	
51	9.58 919	30	9.62 468	36	0.37 532	9.96 451	5	9	5 5 5
52	9.58 949	30	9.62 504	35	0.37 496	9.96 445	6	8	37 36
53	9.58 979	30	9.62 539	35	0.37 461	9.96 440	5	7	0 3.7 3.6 3
54	9.59 009	30	9.62 574	35	0.37 426	9.96 435	5	6	1 11.1 10.8 10
55	9.59 039	30	9.62 609	35	0.37 391	9.96 429	5	5	2 18.5 18.0 17
56	9.59 069	30	9.62 645	36	0.37 355	9.96 424	5	4	3 25.9 25.2 24
57	9.59 098	29	9.62 680	35	0.37 320	9.96 419	5	3	4 33.3 32.4 31
58	9.59 128	30	9.62 715	35	0.37 285	9.96 413	6	2	
59	9.59 158	30	9.62 750	35	0.37 250	9.96 408	5	1	
60	9.59 188	30	9.62 785	35	0.37 215	9.96 403	5	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.59 188	30	9.62 785	35	0.37 215	9.96 403	6	60	
1	9.59 218	29	9.62 820	35	0.37 180	9.96 397	5	59	
2	9.59 247	30	9.62 855	35	0.37 145	9.96 392	5	58	
3	9.59 277	30	9.62 890	35	0.37 110	9.96 387	5	57	36 35 34
4	9.59 307	29	9.62 926	36	0.37 074	9.96 381	6	56	3.6 3.5 3.4
5	9.59 336	29	9.62 961	35	0.37 039	9.96 376	5	55	7.2 7.0 6.8
6	9.59 366	30	9.62 996	35	0.37 004	9.96 370	6	54	10.8 10.5 10.2
7	9.59 396	30	9.63 031	35	0.36 969	9.96 365	5	53	14.4 14.0 13.6
8	9.59 425	29	9.63 066	35	0.36 934	9.96 360	5	52	18.0 17.5 17.0
9	9.59 455	29	9.63 101	35	0.36 899	9.96 354	6	51	21.6 21.0 20.4
10	9.59 484	30	9.63 135	34	0.36 865	9.96 349	5	50	25.2 24.5 23.8
11	9.59 514	29	9.63 170	35	0.36 830	9.96 343	6	49	28.8 28.0 27.2
12	9.59 543	29	9.63 205	35	0.36 795	9.96 338	5	48	32.4 31.5 30.6
13	9.59 573	30	9.63 240	35	0.36 760	9.96 333	6	47	
14	9.59 602	29	9.63 275	35	0.36 725	9.96 327	5	46	
15	9.59 632	30	9.63 310	35	0.36 690	9.96 322	6	45	30 29 28
16	9.59 661	29	9.63 345	35	0.36 655	9.96 316	5	44	3.0 2.9 2.8
17	9.59 690	29	9.63 379	34	0.36 621	9.96 311	6	43	6.0 5.8 5.6
18	9.59 720	30	9.63 414	35	0.36 586	9.96 305	5	42	9.0 8.7 8.4
19	9.59 749	29	9.63 449	35	0.36 551	9.96 300	6	41	12.0 11.6 11.2
20	9.59 778	30	9.63 484	35	0.36 516	9.96 294	5	40	15.0 14.5 14.0
21	9.59 808	29	9.63 519	35	0.36 481	9.96 289	6	39	18.0 17.4 16.8
22	9.59 837	29	9.63 553	34	0.36 447	9.96 284	5	38	21.0 20.3 19.6
23	9.59 866	29	9.63 588	35	0.36 412	9.96 278	6	37	24.0 23.2 22.4
24	9.59 895	29	9.63 623	35	0.36 377	9.96 273	5	36	27.0 26.1 25.2
25	9.59 924	29	9.63 657	34	0.36 343	9.96 267	6	35	
26	9.59 954	30	9.63 692	35	0.36 308	9.96 262	5	34	
27	9.59 983	29	9.63 726	34	0.36 274	9.96 256	6	33	6 5
28	9.60 012	29	9.63 761	35	0.36 239	9.96 251	5	32	0.6 0.5
29	9.60 041	29	9.63 796	35	0.36 204	9.96 245	6	31	1.2 1.0
30	9.60 070	29	9.63 830	34	0.36 170	9.96 240	5	30	1.8 1.5
31	9.60 099	29	9.63 865	35	0.36 135	9.96 234	6	29	2.4 2.0
32	9.60 128	29	9.63 899	34	0.36 101	9.96 229	5	28	3.0 2.5
33	9.60 157	29	9.63 934	35	0.36 066	9.96 223	6	27	3.6 3.0
34	9.60 186	29	9.63 968	34	0.36 032	9.96 218	5	26	4.2 3.5
35	9.60 215	29	9.64 003	35	0.35 997	9.96 212	6	25	4.8 4.0
36	9.60 244	29	9.64 037	34	0.35 963	9.96 207	5	24	5.4 4.5
37	9.60 273	29	9.64 072	35	0.35 928	9.96 201	6	23	
38	9.60 302	29	9.64 106	34	0.35 894	9.96 196	5	22	
39	9.60 331	28	9.64 140	35	0.35 860	9.96 190	6	21	
40	9.60 359	29	9.64 175	34	0.35 825	9.96 185	5	20	
41	9.60 388	29	9.64 209	35	0.35 791	9.96 179	6	19	6 6 6
42	9.60 417	29	9.64 243	34	0.35 757	9.96 174	5	18	36 35 34
43	9.60 446	28	9.64 278	35	0.35 722	9.96 168	6	17	
44	9.60 474	29	9.64 312	34	0.35 688	9.96 162	5	16	3.0 2.9 2.8
45	9.60 503	29	9.64 346	35	0.35 654	9.96 157	6	15	9.0 8.8 8.5
46	9.60 532	29	9.64 381	34	0.35 619	9.96 151	5	14	15.0 14.6 14.2
47	9.60 561	28	9.64 415	35	0.35 585	9.96 146	6	13	21.0 20.4 19.8
48	9.60 589	29	9.64 449	34	0.35 551	9.96 140	5	12	27.0 26.2 25.5
49	9.60 618	28	9.64 483	35	0.35 517	9.96 135	6	11	33.0 32.1 31.2
50	9.60 646	29	9.64 517	34	0.35 483	9.96 129	5	10	
51	9.60 675	29	9.64 552	35	0.35 448	9.96 123	6	9	
	9.60 704	28	9.64 586	34	0.35 414	9.96 118	5	8	5 5
	9.60 732	29	9.64 620	35	0.35 380	9.96 112	6	7	35 34
	9.60 761	29	9.64 654	34	0.35 346	9.96 107	5	6	
	9.60 789	28	9.64 688	35	0.35 312	9.96 101	6	5	3.5 3.4
	9.60 818	29	9.64 722	34	0.35 278	9.96 095	5	4	10.5 10.2
	9.60 846	28	9.64 756	35	0.35 244	9.96 090	6	3	17.5 17.0
	9.60 875	29	9.64 790	34	0.35 210	9.96 084	5	2	24.5 23.8
	9.60 903	28	9.64 824	35	0.35 176	9.96 079	6	1	31.5 30.6
	9.60 931	29	9.64 858	34	0.35 142	9.96 073	5	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

'	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.60 931	29	9.64 858	34	0.35 142	9.96 073	6	60	
1	9.60 960	28	9.64 892	34	0.35 108	9.96 067	5	59	
2	9.60 988	28	9.64 926	34	0.35 074	9.96 062	5	58	
3	9.61 016	28	9.64 960	34	0.35 040	9.96 056	6	57	
4	9.61 045	29	9.64 994	34	0.35 006	9.96 050	5	56	34 33
5	9.61 073	28	9.65 028	34	0.34 972	9.96 045	5	55	I 3.4 3.3
6	9.61 101	28	9.65 062	34	0.34 938	9.96 039	6	54	2 6.8 6.6
7	9.61 129	28	9.65 096	34	0.34 904	9.96 034	5	53	3 10.2 9.9
8	9.61 158	29	9.65 130	34	0.34 870	9.96 028	6	52	4 13.6 13.2
9	9.61 186	28	9.65 164	34	0.34 836	9.96 022	5	51	5 17.0 16.5
10	9.61 214	28	9.65 197	33	0.34 803	9.96 017	6	50	6 20.4 19.8
11	9.61 242	28	9.65 231	34	0.34 769	9.96 011	5	49	7 23.8 23.1
12	9.61 270	28	9.65 265	34	0.34 735	9.96 005	6	48	8 27.2 26.4
13	9.61 298	28	9.65 299	34	0.34 701	9.96 000	5	47	9 30.6 29.7
14	9.61 326	28	9.65 333	34	0.34 667	9.95 994	6	46	
15	9.61 354	28	9.65 366	33	0.34 634	9.95 988	5	45	
16	9.61 382	28	9.65 400	34	0.34 600	9.95 982	6	44	
17	9.61 411	29	9.65 434	34	0.34 566	9.95 977	5	43	29 28 27
18	9.61 438	27	9.65 467	33	0.34 533	9.95 971	6	42	I 2.9 2.8 2.7
19	9.61 466	28	9.65 501	34	0.34 499	9.95 965	5	41	2 5.8 5.6 5.4
20	9.61 494	28	9.65 535	33	0.34 465	9.95 960	6	40	3 8.7 8.4 8.1
21	9.61 522	28	9.65 568	34	0.34 432	9.95 954	5	39	4 11.6 11.2 10.8
22	9.61 550	28	9.65 602	34	0.34 398	9.95 948	6	38	5 14.5 14.0 13.5
23	9.61 578	28	9.65 636	34	0.34 364	9.95 942	5	37	6 17.4 16.8 16.2
24	9.61 606	28	9.65 669	33	0.34 331	9.95 937	6	36	7 20.3 19.6 18.9
25	9.61 634	28	9.65 703	34	0.34 297	9.95 931	5	35	8 23.2 22.4 21.6
26	9.61 662	28	9.65 736	33	0.34 264	9.95 925	6	34	9 26.1 25.2 24.3
27	9.61 689	27	9.65 770	34	0.34 230	9.95 920	5	33	
28	9.61 717	28	9.65 803	33	0.34 197	9.95 914	6	32	
29	9.61 745	28	9.65 837	34	0.34 163	9.95 908	5	31	
30	9.61 773	27	9.65 870	33	0.34 130	9.95 902	6	30	
31	9.61 800	28	9.65 904	34	0.34 096	9.95 897	5	29	6 5
32	9.61 828	28	9.65 937	33	0.34 063	9.95 891	6	28	I 0.6 0.5
33	9.61 856	28	9.65 971	34	0.34 029	9.95 885	5	27	2 1.2 1.0
34	9.61 883	27	9.66 004	33	0.33 996	9.95 879	6	26	3 1.8 1.5
35	9.61 911	28	9.66 038	34	0.33 962	9.95 873	5	25	4 2.4 2.0
36	9.61 939	28	9.66 071	33	0.33 929	9.95 868	6	24	5 3.0 2.5
37	9.61 966	27	9.66 104	33	0.33 896	9.95 862	5	23	6 3.6 3.0
38	9.61 994	28	9.66 138	34	0.33 862	9.95 856	6	22	7 4.2 3.5
39	9.62 021	27	9.66 171	33	0.33 829	9.95 850	5	21	8 4.8 4.0
40	9.62 049	28	9.66 204	33	0.33 796	9.95 844	6	20	9 5.4 4.5
41	9.62 076	27	9.66 238	34	0.33 762	9.95 839	5	19	
42	9.62 104	28	9.66 271	33	0.33 729	9.95 833	6	18	
43	9.62 131	27	9.66 304	33	0.33 696	9.95 827	5	17	
44	9.62 159	28	9.66 337	34	0.33 663	9.95 821	6	16	
45	9.62 186	27	9.66 371	33	0.33 629	9.95 815	5	15	
46	9.62 214	28	9.66 404	33	0.33 596	9.95 810	6	14	
47	9.62 241	27	9.66 437	33	0.33 563	9.95 804	5	13	
48	9.62 268	27	9.66 470	33	0.33 530	9.95 798	6	12	
49	9.62 296	28	9.66 503	34	0.33 497	9.95 792	5	11	
50	9.62 323	27	9.66 537	33	0.33 463	9.95 786	6	10	
51	9.62 350	27	9.66 570	33	0.33 430	9.95 780	5	9	6 6 5
52	9.62 377	28	9.66 603	33	0.33 397	9.95 775	6	8	I 2.8 2.8 3.4
53	9.62 405	27	9.66 636	33	0.33 364	9.95 769	5	7	8.5 8.2 10
54	9.62 432	27	9.66 669	33	0.33 331	9.95 763	6	6	14.2 13.8 11
55	9.62 459	27	9.66 702	33	0.33 298	9.95 757	5	5	19.8 19.2 21
56	9.62 486	27	9.66 735	33	0.33 265	9.95 751	6	4	25.5 24.8 30
57	9.62 513	27	9.66 768	33	0.33 232	9.95 745	5	3	31.2 30.2 -
58	9.62 541	28	9.66 801	33	0.33 199	9.95 739	6	2	
59	9.62 568	27	9.66 834	33	0.33 166	9.95 733	5	1	
60	9.62 595	27	9.66 867	33	0.33 133	9.95 728	6	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	'	P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.	P. P.
0	9.62 593	27	9.66 867	33	0.33 133	9.95 728	6	60
1	9.62 622	27	9.66 900	33	0.33 100	9.95 722	6	59
2	9.62 649	27	9.66 933	33	0.33 067	9.95 716	6	58
3	9.62 676	27	9.66 966	33	0.33 034	9.95 710	6	57
4	9.62 703	27	9.66 999	33	0.33 001	9.95 704	6	56
5	9.62 730	27	9.67 032	33	0.32 968	9.95 698	6	55
6	9.62 757	27	9.67 065	33	0.32 935	9.95 692	6	54
7	9.62 784	27	9.67 098	33	0.32 902	9.95 686	6	53
8	9.62 811	27	9.67 131	32	0.32 869	9.95 680	6	52
9	9.62 838	27	9.67 163	33	0.32 837	9.95 674	6	51
10	9.62 865	27	9.67 196	33	0.32 804	9.95 668	6	50
11	9.62 892	27	9.67 229	33	0.32 771	9.95 663	5	49
12	9.62 918	26	9.67 262	33	0.32 738	9.95 657	6	48
13	9.62 945	27	9.67 295	33	0.32 705	9.95 651	6	47
14	9.62 972	27	9.67 327	32	0.32 673	9.95 645	6	46
15	9.62 999	27	9.67 360	33	0.32 640	9.95 639	6	45
16	9.63 026	26	9.67 393	33	0.32 607	9.95 633	6	44
17	9.63 052	27	9.67 426	32	0.32 574	9.95 627	6	43
18	9.63 079	27	9.67 458	32	0.32 542	9.95 621	6	42
19	9.63 106	27	9.67 491	33	0.32 509	9.95 615	6	41
20	9.63 133	26	9.67 524	32	0.32 476	9.95 609	6	40
21	9.63 159	27	9.67 556	33	0.32 444	9.95 603	6	39
22	9.63 186	27	9.67 589	33	0.32 411	9.95 597	6	38
23	9.63 213	26	9.67 622	33	0.32 378	9.95 591	6	37
24	9.63 239	27	9.67 654	32	0.32 346	9.95 585	6	36
25	9.63 266	26	9.67 687	33	0.32 313	9.95 579	6	35
26	9.63 292	27	9.67 719	32	0.32 281	9.95 573	6	34
27	9.63 319	26	9.67 752	33	0.32 248	9.95 567	6	33
28	9.63 345	27	9.67 785	32	0.32 215	9.95 561	6	32
29	9.63 372	26	9.67 817	33	0.32 183	9.95 555	6	31
30	9.63 398	27	9.67 850	32	0.32 150	9.95 549	6	30
31	9.63 425	26	9.67 882	33	0.32 118	9.95 543	6	29
32	9.63 451	27	9.67 915	32	0.32 085	9.95 537	6	28
33	9.63 478	26	9.67 947	33	0.32 053	9.95 531	6	27
34	9.63 504	27	9.67 980	32	0.32 020	9.95 525	6	26
35	9.63 531	26	9.68 012	32	0.31 988	9.95 519	6	25
36	9.63 557	26	9.68 044	32	0.31 956	9.95 513	6	24
37	9.63 583	27	9.68 077	33	0.31 923	9.95 507	7	23
38	9.63 610	26	9.68 109	33	0.31 891	9.95 500	6	22
39	9.63 636	26	9.68 142	32	0.31 858	9.95 494	6	21
40	9.63 662	27	9.68 174	32	0.31 826	9.95 488	6	20
41	9.63 689	26	9.68 206	33	0.31 794	9.95 482	6	19
42	9.63 715	26	9.68 239	32	0.31 761	9.95 476	6	18
43	9.63 741	26	9.68 271	32	0.31 729	9.95 470	6	17
44	9.63 767	27	9.68 303	32	0.31 697	9.95 464	6	16
45	9.63 794	26	9.68 336	32	0.31 664	9.95 458	6	15
46	9.63 820	26	9.68 368	32	0.31 632	9.95 452	6	14
47	9.63 846	26	9.68 400	32	0.31 600	9.95 446	6	13
48	9.63 872	26	9.68 432	32	0.31 568	9.95 440	6	12
49	9.63 898	26	9.68 465	32	0.31 535	9.95 434	6	11
50	9.63 924	26	9.68 497	32	0.31 503	9.95 427	7	10
	9.63 950	26	9.68 529	32	0.31 471	9.95 421	6	9
	9.63 976	26	9.68 561	32	0.31 439	9.95 415	6	8
	9.64 002	26	9.68 593	32	0.31 407	9.95 409	6	7
	9.64 028	26	9.68 626	33	0.31 374	9.95 403	6	6
	9.64 054	26	9.68 658	32	0.31 342	9.95 397	6	5
	9.64 080	26	9.68 690	32	0.31 310	9.95 391	6	4
	9.64 106	26	9.68 722	32	0.31 278	9.95 384	7	3
	9.64 132	26	9.68 754	32	0.31 246	9.95 378	6	2
	9.64 158	26	9.68 786	32	0.31 214	9.95 372	6	1
()	9.64 184	26	9.68 818	32	0.31 182	9.95 366	6	0
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.64 184	26	9.68 818	32	0.31 182	9.95 366	6	60	
1	9.64 210	26	9.68 850	32	0.31 150	9.95 360	6	59	
2	9.64 236	26	9.68 882	32	0.31 118	9.95 354	6	58	
3	9.64 262	26	9.68 914	32	0.31 086	9.95 348	6	57	
4	9.64 288	26	9.68 946	32	0.31 054	9.95 341	7	56	32 31
5	9.64 313	25	9.68 978	32	0.31 022	9.95 335	6	55	1 3.2 3.1
6	9.64 339	26	9.69 010	32	0.30 990	9.95 329	6	54	2 6.4 6.2
7	9.64 365	26	9.69 042	32	0.30 958	9.95 323	6	53	3 9.6 9.3
8	9.64 391	26	9.69 074	32	0.30 926	9.95 317	6	52	4 12.8 12.4
9	9.64 417	25	9.69 106	32	0.30 894	9.95 310	7	51	5 16.0 15.5
10	9.64 442	26	9.69 138	32	0.30 862	9.95 304	6	50	6 19.2 18.6
11	9.64 468	26	9.69 170	32	0.30 830	9.95 298	6	49	7 22.4 21.7
12	9.64 494	25	9.69 202	32	0.30 798	9.95 292	6	48	8 25.6 24.8
13	9.64 519	26	9.69 234	32	0.30 766	9.95 286	6	47	9 28.8 27.9
14	9.64 545	26	9.69 266	32	0.30 734	9.95 279	7	46	
15	9.64 571	25	9.69 298	32	0.30 702	9.95 273	6	45	
16	9.64 596	26	9.69 329	31	0.30 671	9.95 267	6	44	
17	9.64 622	26	9.69 361	32	0.30 639	9.95 261	6	43	26 25 24
18	9.64 647	25	9.69 393	32	0.30 607	9.95 254	7	42	1 2.6 2.5 2.4
19	9.64 673	26	9.69 425	32	0.30 575	9.95 248	6	41	2 5.2 5.0 4.8
20	9.64 698	25	9.69 457	32	0.30 543	9.95 242	6	40	3 7.8 7.5 7.2
21	9.64 724	26	9.69 488	31	0.30 512	9.95 236	6	39	4 10.4 10.0 9.6
22	9.64 749	25	9.69 520	32	0.30 480	9.95 229	7	38	5 13.0 12.5 12.0
23	9.64 775	26	9.69 552	32	0.30 448	9.95 223	6	37	6 15.6 15.0 14.4
24	9.64 800	25	9.69 584	32	0.30 416	9.95 217	6	36	7 18.2 17.5 16.8
25	9.64 826	26	9.69 615	31	0.30 385	9.95 211	6	35	8 20.8 20.0 19.2
26	9.64 851	25	9.69 647	32	0.30 353	9.95 204	7	34	9 23.4 22.5 21.6
27	9.64 877	26	9.69 679	32	0.30 321	9.95 198	6	33	
28	9.64 902	25	9.69 710	31	0.30 290	9.95 192	6	32	
29	9.64 927	26	9.69 742	32	0.30 258	9.95 185	7	31	
30	9.64 953	25	9.69 774	32	0.30 226	9.95 179	6	30	7 6
31	9.64 978	26	9.69 805	31	0.30 195	9.95 173	6	29	1 0.7 0.6
32	9.65 003	25	9.69 837	32	0.30 163	9.95 167	6	28	2 1.4 1.2
33	9.65 029	26	9.69 868	31	0.30 132	9.95 160	7	27	3 2.1 1.8
34	9.65 054	25	9.69 900	32	0.30 100	9.95 154	6	26	4 2.8 2.4
35	9.65 079	26	9.69 932	32	0.30 068	9.95 148	6	25	5 3.5 3.0
36	9.65 104	25	9.69 963	31	0.30 037	9.95 141	7	24	6 4.2 3.6
37	9.65 130	26	9.69 995	32	0.30 005	9.95 135	6	23	7 4.9 4.2
38	9.65 155	25	9.70 026	32	0.29 974	9.95 129	6	22	8 5.6 4.8
39	9.65 180	26	9.70 058	32	0.29 942	9.95 122	7	21	9 6.3 5.4
40	9.65 205	25	9.70 089	31	0.29 911	9.95 116	6	20	
41	9.65 230	26	9.70 121	32	0.29 879	9.95 110	6	19	
42	9.65 255	25	9.70 152	31	0.29 848	9.95 103	7	18	
43	9.65 281	26	9.70 184	32	0.29 816	9.95 097	6	17	
44	9.65 306	25	9.70 215	32	0.29 785	9.95 090	7	16	
45	9.65 331	26	9.70 247	31	0.29 753	9.95 084	6	15	
46	9.65 356	25	9.70 278	32	0.29 722	9.95 078	6	14	
47	9.65 381	26	9.70 309	31	0.29 691	9.95 071	7	13	
48	9.65 406	25	9.70 341	32	0.29 659	9.95 065	6	12	
49	9.65 431	26	9.70 372	31	0.29 628	9.95 059	7	11	
50	9.65 456	25	9.70 404	32	0.29 596	9.95 052	6	10	
51	9.65 481	26	9.70 435	31	0.29 565	9.95 046	7	9	
52	9.65 506	25	9.70 466	32	0.29 534	9.95 039	6	8	
53	9.65 531	26	9.70 498	31	0.29 502	9.95 033	7	7	
54	9.65 556	25	9.70 529	32	0.29 471	9.95 027	6	6	
55	9.65 580	26	9.70 560	31	0.29 440	9.95 020	7	5	
56	9.65 605	25	9.70 592	32	0.29 408	9.95 014	6	4	
57	9.65 630	26	9.70 623	31	0.29 377	9.95 007	7	3	
58	9.65 655	25	9.70 654	32	0.29 346	9.95 001	6	2	
59	9.65 680	26	9.70 685	31	0.29 315	9.94 995	7	1	
60	9.65 705	25	9.70 717	32	0.29 283	9.94 988	6	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.65 705	24	9.70 717	31	0.29 283	9.94 988	6	60	
1	9.65 729	25	9.70 748	31	0.29 252	9.94 982	7	59	
2	9.65 754	25	9.70 779	31	0.29 221	9.94 975	7	58	
3	9.65 779	25	9.70 810	31	0.29 190	9.94 969	7	57	
4	9.65 804	24	9.70 841	31	0.29 159	9.94 962	7	56	32 31 30
5	9.65 828	25	9.70 873	31	0.29 127	9.94 956	7	55	1 3.2 3.1 3.0
6	9.65 853	25	9.70 904	31	0.29 096	9.94 949	7	54	2 6.4 6.2 6.0
7	9.65 878	25	9.70 935	31	0.29 065	9.94 943	7	53	3 9.6 9.3 9.0
8	9.65 902	25	9.70 966	31	0.29 034	9.94 936	7	52	4 12.8 12.4 12.0
9	9.65 927	25	9.70 997	31	0.29 003	9.94 930	7	51	5 16.0 15.5 15.0
10	9.65 952	24	9.71 028	31	0.28 972	9.94 923	7	50	6 19.2 18.6 18.0
11	9.65 976	25	9.71 059	31	0.28 941	9.94 917	7	49	7 22.4 21.7 21.0
12	9.66 001	24	9.71 090	31	0.28 910	9.94 911	7	48	8 25.6 24.8 24.0
13	9.66 025	25	9.71 121	31	0.28 879	9.94 904	7	47	9 28.8 27.9 27.0
14	9.66 050	25	9.71 153	32	0.28 847	9.94 898	7	46	
15	9.66 075	25	9.71 184	31	0.28 816	9.94 891	7	45	
16	9.66 099	24	9.71 215	31	0.28 785	9.94 885	7	44	
17	9.66 124	24	9.71 246	31	0.28 754	9.94 878	7	43	25 24 23
18	9.66 148	25	9.71 277	31	0.28 723	9.94 871	7	42	1 2.5 2.4 2.3
19	9.66 173	24	9.71 308	31	0.28 692	9.94 865	7	41	2 5.0 4.8 4.6
20	9.66 197	24	9.71 339	31	0.28 661	9.94 858	7	40	3 7.5 7.2 6.9
21	9.66 221	25	9.71 370	31	0.28 630	9.94 852	7	39	4 10.0 9.6 9.2
22	9.66 246	24	9.71 401	31	0.28 599	9.94 845	7	38	5 12.5 12.0 11.5
23	9.66 270	25	9.71 431	30	0.28 569	9.94 839	7	37	6 15.0 14.4 13.8
24	9.66 295	24	9.71 462	31	0.28 538	9.94 832	7	36	7 17.5 16.8 16.1
25	9.66 319	24	9.71 493	31	0.28 507	9.94 826	7	35	8 20.0 19.2 18.4
26	9.66 343	25	9.71 524	31	0.28 476	9.94 819	7	34	9 22.5 21.6 20.7
27	9.66 368	24	9.71 555	31	0.28 445	9.94 813	7	33	
28	9.66 392	24	9.71 586	31	0.28 414	9.94 806	7	32	
29	9.66 416	25	9.71 617	31	0.28 383	9.94 799	7	31	
30	9.66 441	24	9.71 648	31	0.28 352	9.94 793	7	30	7 6
31	9.66 465	24	9.71 679	30	0.28 321	9.94 786	7	29	1 0.7 0.6
32	9.66 489	24	9.71 709	31	0.28 291	9.94 780	7	28	2 1.4 1.2
33	9.66 513	24	9.71 740	31	0.28 260	9.94 773	7	27	3 2.1 1.8
34	9.66 537	25	9.71 771	31	0.28 229	9.94 767	7	26	4 2.8 2.4
35	9.66 562	24	9.71 802	31	0.28 198	9.94 760	7	25	5 3.5 3.0
36	9.66 586	24	9.71 833	31	0.28 167	9.94 753	7	24	6 4.2 3.6
37	9.66 610	24	9.71 863	30	0.28 137	9.94 747	7	23	7 4.9 4.2
38	9.66 634	24	9.71 894	31	0.28 106	9.94 740	7	22	8 5.6 4.8
39	9.66 658	24	9.71 925	31	0.28 075	9.94 734	7	21	9 6.3 5.4
40	9.66 682	24	9.71 955	30	0.28 045	9.94 727	7	20	
41	9.66 706	25	9.71 986	31	0.28 014	9.94 720	7	19	
42	9.66 731	24	9.72 017	31	0.27 983	9.94 714	7	18	
43	9.66 755	24	9.72 048	31	0.27 952	9.94 707	7	17	
44	9.66 779	24	9.72 078	30	0.27 922	9.94 700	7	16	
45	9.66 803	24	9.72 109	31	0.27 891	9.94 694	7	15	
46	9.66 827	24	9.72 140	31	0.27 860	9.94 687	7	14	
47	9.66 851	24	9.72 170	30	0.27 830	9.94 680	7	13	7 6 6
48	9.66 875	24	9.72 201	31	0.27 799	9.94 674	7	12	30 31 30
49	9.66 899	23	9.72 231	31	0.27 769	9.94 667	7	11	
50	9.66 922	24	9.72 262	31	0.27 738	9.94 660	7	10	
51	9.66 946	24	9.72 293	30	0.27 707	9.94 654	7	9	0 2.1 2.6 2.5
52	9.66 970	24	9.72 323	31	0.27 677	9.94 647	7	8	1 6.4 7.8 7.5
53	9.66 994	24	9.72 354	31	0.27 646	9.94 640	7	7	2 10.7 12.9 12.5
54	9.67 018	24	9.72 384	30	0.27 616	9.94 634	7	6	3 15.0 18.1 17.5
55	9.67 042	24	9.72 415	31	0.27 585	9.94 627	7	5	4 19.3 23.2 22.5
56	9.67 066	24	9.72 445	30	0.27 555	9.94 620	7	4	5 23.6 28.4 27.5
57	9.67 090	24	9.72 476	31	0.27 524	9.94 614	7	3	6 27.9 — —
58	9.67 113	23	9.72 506	30	0.27 494	9.94 607	7	2	
59	9.67 137	24	9.72 537	31	0.27 463	9.94 600	7	1	
60	9.67 161	24	9.72 567	30	0.27 433	9.94 593	7	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.67 161	24	9.72 567	31	0.27 433	9.94 593	6	60	
1	9.67 185	23	9.72 598	30	0.27 402	9.94 587	7	59	
2	9.67 208	23	9.72 628	31	0.27 372	9.94 580	7	58	
3	9.67 232	24	9.72 659	30	0.27 341	9.94 573	6	57	
4	9.67 256	24	9.72 689	30	0.27 311	9.94 567	7	56	31 30 29
5	9.67 280	24	9.72 720	31	0.27 280	9.94 560	7	55	1 3.1 3.0 2.9
6	9.67 303	23	9.72 750	30	0.27 250	9.94 553	7	54	2 6.2 6.0 5.8
7	9.67 327	24	9.72 780	30	0.27 220	9.94 546	6	53	3 9.3 9.0 8.7
8	9.67 350	23	9.72 811	31	0.27 189	9.94 540	7	52	4 12.4 12.0 11.6
9	9.67 374	24	9.72 841	30	0.27 159	9.94 533	7	51	5 15.5 15.0 14.5
10	9.67 398	24	9.72 872	31	0.27 128	9.94 526	7	50	6 18.6 18.0 17.4
11	9.67 421	23	9.72 902	30	0.27 098	9.94 519	7	49	7 21.7 21.0 20.3
12	9.67 445	24	9.72 932	31	0.27 068	9.94 513	6	48	8 24.8 24.0 23.2
13	9.67 468	23	9.72 963	30	0.27 037	9.94 506	7	47	9 27.9 27.0 26.1
14	9.67 492	24	9.72 993	30	0.27 007	9.94 499	7	46	
15	9.67 515	23	9.73 023	30	0.26 977	9.94 492	7	45	
16	9.67 539	24	9.73 054	31	0.26 946	9.94 485	6	44	
17	9.67 562	23	9.73 084	30	0.26 916	9.94 479	7	43	24 23 22
18	9.67 586	24	9.73 114	30	0.26 886	9.94 472	7	42	1 2.4 2.3 2.2
19	9.67 609	23	9.73 144	31	0.26 856	9.94 465	7	41	2 4.8 4.6 4.4
20	9.67 633	24	9.73 175	30	0.26 825	9.94 458	7	40	3 7.2 6.9 6.6
21	9.67 656	23	9.73 205	30	0.26 795	9.94 451	6	39	4 9.6 9.2 8.8
22	9.67 680	24	9.73 235	30	0.26 765	9.94 445	7	38	5 12.0 11.5 11.0
23	9.67 703	23	9.73 265	30	0.26 735	9.94 438	7	37	6 14.4 13.8 13.2
24	9.67 726	23	9.73 295	31	0.26 705	9.94 431	7	36	7 16.8 16.1 15.4
25	9.67 750	24	9.73 326	30	0.26 674	9.94 424	7	35	8 19.2 18.4 17.6
26	9.67 773	23	9.73 356	30	0.26 644	9.94 417	7	34	9 21.6 20.7 19.8
27	9.67 796	23	9.73 386	30	0.26 614	9.94 410	6	33	
28	9.67 820	24	9.73 416	30	0.26 584	9.94 404	7	32	
29	9.67 843	23	9.73 446	30	0.26 554	9.94 397	7	31	
30	9.67 866	23	9.73 476	31	0.26 524	9.94 390	7	30	7 6
31	9.67 890	24	9.73 507	30	0.26 493	9.94 383	7	29	1 0.7 0.6
32	9.67 913	23	9.73 537	30	0.26 463	9.94 376	7	28	2 1.4 1.2
33	9.67 936	23	9.73 567	30	0.26 433	9.94 369	7	27	3 2.1 1.8
34	9.67 959	23	9.73 597	30	0.26 403	9.94 362	7	26	4 2.8 2.4
35	9.67 982	23	9.73 627	30	0.26 373	9.94 355	7	25	5 3.5 3.0
36	9.68 006	24	9.73 657	30	0.26 343	9.94 349	6	24	6 4.2 3.6
37	9.68 029	23	9.73 687	30	0.26 313	9.94 342	7	23	7 4.9 4.2
38	9.68 052	23	9.73 717	30	0.26 283	9.94 335	7	22	8 5.6 4.8
39	9.68 075	23	9.73 747	30	0.26 253	9.94 328	7	21	9 6.3 5.4
40	9.68 098	23	9.73 777	30	0.26 223	9.94 321	7	20	
41	9.68 121	23	9.73 807	30	0.26 193	9.94 314	7	19	
42	9.68 144	23	9.73 837	30	0.26 163	9.94 307	7	18	
43	9.68 167	23	9.73 867	30	0.26 133	9.94 300	7	17	
44	9.68 190	23	9.73 897	30	0.26 103	9.94 293	7	16	
45	9.68 213	23	9.73 927	30	0.26 073	9.94 286	7	15	
46	9.68 237	24	9.73 957	30	0.26 043	9.94 279	7	14	
47	9.68 260	23	9.73 987	30	0.26 013	9.94 273	6	13	
48	9.68 283	23	9.74 017	30	0.25 983	9.94 266	7	12	
49	9.68 305	22	9.74 047	30	0.25 953	9.94 259	7	11	7 6 6
50	9.68 328	23	9.74 077	30	0.25 923	9.94 252	7	10	31 31 30
51	9.68 351	23	9.74 107	30	0.25 893	9.94 245	7	9	0 2.2 2.6 2.5
52	9.68 374	23	9.74 137	29	0.25 863	9.94 238	7	8	1 6.6 7.8 7
53	9.68 397	23	9.74 166	30	0.25 834	9.94 231	7	7	2 11.1 12.9 12
54	9.68 420	23	9.74 196	30	0.25 804	9.94 224	7	6	3 15.5 18.1 17
55	9.68 443	23	9.74 226	30	0.25 774	9.94 217	7	5	4 19.9 23.2 22
56	9.68 466	23	9.74 256	30	0.25 744	9.94 210	7	4	5 24.4 28.4 27
57	9.68 489	23	9.74 286	30	0.25 714	9.94 203	7	3	6 28.8 — —
58	9.68 512	22	9.74 316	29	0.25 684	9.94 196	7	2	
59	9.68 534	23	9.74 345	30	0.25 655	9.94 189	7	1	
60	9.68 557	23	9.74 375	30	0.25 625	9.94 182	7	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.68 557	23	9.74 375	30	0.25 625	9.94 182	7	60	
1	9.68 580	23	9.74 405	30	0.25 595	9.94 175	7	59	
2	9.68 603	22	9.74 435	30	0.25 565	9.94 168	7	58	
3	9.68 625	23	9.74 465	29	0.25 535	9.94 161	7	57	30 29 23
4	9.68 648	23	9.74 494	30	0.25 506	9.94 154	7	56	1 3.0 2.9 2.3
5	9.68 671	23	9.74 524	30	0.25 476	9.94 147	7	55	2 6.0 5.8 4.6
6	9.68 694	22	9.74 554	29	0.25 446	9.94 140	7	54	3 9.0 8.7 6.9
7	9.68 716	23	9.74 583	30	0.25 417	9.94 133	7	53	4 12.0 11.6 9.2
8	9.68 739	23	9.74 613	30	0.25 387	9.94 126	7	52	5 15.0 14.5 11.5
9	9.68 762	22	9.74 643	30	0.25 357	9.94 119	7	51	6 18.0 17.4 13.8
10	9.68 784	23	9.74 673	29	0.25 327	9.94 112	7	50	7 21.0 20.3 16.1
11	9.68 807	22	9.74 702	30	0.25 298	9.94 105	7	49	8 24.0 23.2 18.4
12	9.68 829	23	9.74 732	30	0.25 268	9.94 098	7	48	9 27.0 26.1 20.7
13	9.68 852	23	9.74 762	29	0.25 238	9.94 090	7	47	
14	9.68 875	22	9.74 791	30	0.25 209	9.94 083	7	46	
15	9.68 897	23	9.74 821	30	0.25 179	9.94 076	7	45	
16	9.68 920	22	9.74 851	29	0.25 149	9.94 069	7	44	22 8 7
17	9.68 942	23	9.74 880	30	0.25 120	9.94 062	7	43	1 2.2 0.8 0.7
18	9.68 965	22	9.74 910	30	0.25 090	9.94 055	7	42	2 4.4 1.6 1.4
19	9.68 987	23	9.74 939	29	0.25 061	9.94 048	7	41	3 6.6 2.4 2.1
20	9.69 010	22	9.74 969	29	0.25 031	9.94 041	7	40	4 8.8 3.2 2.8
21	9.69 032	23	9.74 998	30	0.25 002	9.94 034	7	39	5 11.0 4.0 3.5
22	9.69 055	22	9.75 028	30	0.24 972	9.94 027	7	38	6 13.2 4.8 4.2
23	9.69 077	23	9.75 058	29	0.24 942	9.94 020	7	37	7 15.4 5.6 4.9
24	9.69 100	22	9.75 087	30	0.24 913	9.94 012	7	36	8 17.6 6.4 5.6
25	9.69 122	22	9.75 117	29	0.24 883	9.94 005	7	35	9 19.8 7.2 6.3
26	9.69 144	23	9.75 146	30	0.24 854	9.93 998	7	34	
27	9.69 167	22	9.75 176	29	0.24 824	9.93 991	7	33	
28	9.69 189	23	9.75 205	30	0.24 795	9.93 984	7	32	
29	9.69 212	22	9.75 235	29	0.24 765	9.93 977	7	31	
30	9.69 234	22	9.75 264	30	0.24 736	9.93 970	7	30	
31	9.69 256	23	9.75 294	29	0.24 706	9.93 963	7	29	
32	9.69 279	22	9.75 323	30	0.24 677	9.93 955	7	28	
33	9.69 301	22	9.75 353	29	0.24 647	9.93 948	7	27	
34	9.69 323	22	9.75 382	29	0.24 618	9.93 941	7	26	8 8
35	9.69 345	23	9.75 411	30	0.24 589	9.93 934	7	25	30 29
36	9.69 368	22	9.75 441	29	0.24 559	9.93 927	7	24	
37	9.69 390	22	9.75 470	30	0.24 530	9.93 920	7	23	
38	9.69 412	22	9.75 500	29	0.24 500	9.93 912	7	22	
39	9.69 434	22	9.75 529	29	0.24 471	9.93 905	7	21	
40	9.69 456	23	9.75 558	30	0.24 442	9.93 898	7	20	
41	9.69 479	22	9.75 588	29	0.24 412	9.93 891	7	19	
42	9.69 501	22	9.75 617	30	0.24 383	9.93 884	7	18	
43	9.69 523	22	9.75 647	29	0.24 353	9.93 876	7	17	
44	9.69 545	22	9.75 676	29	0.24 324	9.93 869	7	16	
45	9.69 567	22	9.75 705	30	0.24 295	9.93 862	7	15	
46	9.69 589	22	9.75 735	29	0.24 265	9.93 855	7	14	
47	9.69 611	22	9.75 764	29	0.24 236	9.93 847	7	13	
48	9.69 633	22	9.75 793	29	0.24 207	9.93 840	7	12	
49	9.69 655	22	9.75 822	30	0.24 178	9.93 833	7	11	
50	9.69 677	22	9.75 852	29	0.24 148	9.93 826	7	10	
51	9.69 699	22	9.75 881	29	0.24 119	9.93 819	7	9	
52	9.69 721	22	9.75 910	29	0.24 090	9.93 811	7	8	
53	9.69 743	22	9.75 939	30	0.24 061	9.93 804	7	7	
54	9.69 765	22	9.75 969	29	0.24 031	9.93 797	7	6	
55	9.69 787	22	9.75 998	29	0.24 002	9.93 789	7	5	
56	9.69 809	22	9.76 027	29	0.23 973	9.93 782	7	4	
57	9.69 831	22	9.76 056	30	0.23 944	9.93 775	7	3	
58	9.69 853	22	9.76 086	29	0.23 914	9.93 768	7	2	
59	9.69 875	22	9.76 115	29	0.23 885	9.93 760	7	1	
60	9.69 897	22	9.76 144	29	0.23 856	9.93 753	7	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.69 897	22	9.76 144	29	0.23 856	9.93 753	7	60	
1	9.69 919	22	9.76 173	29	0.23 827	9.93 746	8	59	
2	9.69 941	22	9.76 202	29	0.23 798	9.93 738	7	58	
3	9.69 963	21	9.76 231	30	0.23 769	9.93 731	7	57	
4	9.69 984	22	9.76 261	29	0.23 739	9.93 724	7	56	30 29 28
5	9.70 006	22	9.76 290	29	0.23 710	9.93 717	7	55	1 3.0 2.9 2.8
6	9.70 028	22	9.76 319	29	0.23 681	9.93 709	7	54	2 6.0 5.8 5.6
7	9.70 050	22	9.76 348	29	0.23 652	9.93 702	7	53	3 9.0 8.7 8.4
8	9.70 072	21	9.76 377	29	0.23 623	9.93 695	7	52	4 12.0 11.6 11.2
9	9.70 093	22	9.76 406	29	0.23 594	9.93 687	7	51	5 15.0 14.5 14.0
10	9.70 115	22	9.76 435	29	0.23 565	9.93 680	7	50	6 18.0 17.4 16.8
11	9.70 137	22	9.76 464	29	0.23 536	9.93 673	7	49	7 21.0 20.3 19.6
12	9.70 159	21	9.76 493	29	0.23 507	9.93 665	7	48	8 24.0 23.2 22.4
13	9.70 180	22	9.76 522	29	0.23 478	9.93 658	7	47	9 27.0 26.1 25.2
14	9.70 202	22	9.76 551	29	0.23 449	9.93 650	7	46	
15	9.70 224	21	9.76 580	29	0.23 420	9.93 643	7	45	
16	9.70 245	22	9.76 609	30	0.23 391	9.93 636	7	44	
17	9.70 267	21	9.76 639	29	0.23 361	9.93 628	7	43	22 21
18	9.70 288	22	9.76 668	29	0.23 332	9.93 621	7	42	1 2.2 2.1
19	9.70 310	22	9.76 697	28	0.23 303	9.93 614	7	41	2 4.4 4.2
20	9.70 332	21	9.76 725	29	0.23 275	9.93 606	7	40	3 6.6 6.3
21	9.70 353	22	9.76 754	29	0.23 246	9.93 599	7	39	4 8.8 8.4
22	9.70 375	21	9.76 783	29	0.23 217	9.93 591	7	38	5 11.0 10.5
23	9.70 396	22	9.76 812	29	0.23 188	9.93 584	7	37	6 13.2 12.6
24	9.70 418	21	9.76 841	29	0.23 159	9.93 577	7	36	7 15.4 14.7
25	9.70 439	22	9.76 870	29	0.23 130	9.93 569	7	35	8 17.6 16.8
26	9.70 461	21	9.76 899	29	0.23 101	9.93 562	7	34	9 19.8 18.9
27	9.70 482	22	9.76 928	29	0.23 072	9.93 554	7	33	
28	9.70 504	21	9.76 957	29	0.23 043	9.93 547	7	32	
29	9.70 525	22	9.76 986	29	0.23 014	9.93 539	7	31	
30	9.70 547	21	9.77 015	29	0.22 985	9.93 532	7	30	8 7
31	9.70 568	22	9.77 044	29	0.22 956	9.93 525	7	29	1 0.8 0.7
32	9.70 590	21	9.77 073	28	0.22 927	9.93 517	7	28	2 1.6 1.4
33	9.70 611	22	9.77 101	29	0.22 899	9.93 510	7	27	3 2.4 2.1
34	9.70 633	21	9.77 130	29	0.22 870	9.93 502	7	26	4 3.2 2.8
35	9.70 654	21	9.77 159	29	0.22 841	9.93 495	7	25	5 4.0 3.5
36	9.70 675	22	9.77 188	29	0.22 812	9.93 487	7	24	6 4.8 4.2
37	9.70 697	21	9.77 217	29	0.22 783	9.93 480	7	23	7 5.6 4.9
38	9.70 718	21	9.77 246	28	0.22 754	9.93 472	7	22	8 6.4 5.6
39	9.70 739	22	9.77 274	29	0.22 726	9.93 465	7	21	9 7.2 6.3
40	9.70 761	21	9.77 303	29	0.22 697	9.93 457	7	20	
41	9.70 782	21	9.77 332	29	0.22 668	9.93 450	7	19	
42	9.70 803	21	9.77 361	29	0.22 639	9.93 442	7	18	
43	9.70 824	22	9.77 390	28	0.22 610	9.93 435	7	17	
44	9.70 846	21	9.77 418	28	0.22 582	9.93 427	7	16	
45	9.70 867	21	9.77 447	29	0.22 553	9.93 420	7	15	
46	9.70 888	21	9.77 476	29	0.22 524	9.93 412	7	14	
47	9.70 909	22	9.77 505	28	0.22 495	9.93 405	7	13	7 7 7
48	9.70 931	21	9.77 533	29	0.22 467	9.93 397	7	12	30 29 28
49	9.70 952	21	9.77 562	29	0.22 438	9.93 390	7	11	
50	9.70 973	21	9.77 591	28	0.22 409	9.93 382	7	10	0 2.1 2.1 2.0
51	9.70 994	21	9.77 619	29	0.22 381	9.93 375	7	9	1 6.4 6.2 6.0
52	9.71 015	21	9.77 648	29	0.22 352	9.93 367	7	8	2 10.7 10.4 10
53	9.71 036	22	9.77 677	29	0.22 323	9.93 360	7	7	3 15.0 14.5 14
54	9.71 058	21	9.77 706	28	0.22 294	9.93 352	7	6	4 19.3 18.6 18
55	9.71 079	21	9.77 734	29	0.22 266	9.93 344	7	5	5 23.6 22.8 22
56	9.71 100	21	9.77 763	28	0.22 237	9.93 337	7	4	6 27.9 26.9
57	9.71 121	21	9.77 791	29	0.22 209	9.93 329	7	3	
58	9.71 142	21	9.77 820	29	0.22 180	9.93 322	7	2	
59	9.71 163	21	9.77 849	28	0.22 151	9.93 314	7	1	
60	9.71 184		9.77 877		0.22 123	9.93 307	7	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.71 184	21	9.77 877	29	0.22 123	9.93 307	8	60	
1	9.71 205	21	9.77 906	29	0.22 094	9.93 299	8	59	
2	9.71 226	21	9.77 935	28	0.22 065	9.93 291	8	58	
3	9.71 247	21	9.77 963	29	0.22 037	9.93 284	7	57	
4	9.71 268	21	9.77 992	29	0.22 008	9.93 276	8	56	29 28
5	9.71 289	21	9.78 020	28	0.21 980	9.93 269	7	55	1 2.9 2.8
6	9.71 310	21	9.78 049	28	0.21 951	9.93 261	8	54	2 5.8 5.6
7	9.71 331	21	9.78 077	29	0.21 923	9.93 253	8	53	3 8.7 8.4
8	9.71 352	21	9.78 106	29	0.21 894	9.93 246	7	52	4 11.6 11.2
9	9.71 373	20	9.78 135	28	0.21 865	9.93 238	8	51	5 14.5 14.0
10	9.71 393	21	9.78 163	29	0.21 837	9.93 230	8	50	6 17.4 16.8
11	9.71 414	21	9.78 192	28	0.21 808	9.93 223	7	49	7 20.3 19.6
12	9.71 435	21	9.78 220	29	0.21 780	9.93 215	8	48	8 23.2 22.4
13	9.71 456	21	9.78 249	28	0.21 751	9.93 207	8	47	9 26.1 25.2
14	9.71 477	21	9.78 277	29	0.21 723	9.93 200	7	46	
15	9.71 498	21	9.78 306	28	0.21 694	9.93 192	8	45	
16	9.71 519	20	9.78 334	29	0.21 666	9.93 184	8	44	31 30
17	9.71 539	21	9.78 363	28	0.21 637	9.93 177	7	43	1 2.1 2.0
18	9.71 560	21	9.78 391	28	0.21 609	9.93 169	8	42	2 4.2 4.0
19	9.71 581	21	9.78 419	29	0.21 581	9.93 161	8	41	3 6.3 6.0
20	9.71 602	20	9.78 448	28	0.21 552	9.93 154	8	40	4 8.4 8.0
21	9.71 622	21	9.78 476	29	0.21 524	9.93 146	8	39	5 10.5 10.0
22	9.71 643	21	9.78 505	28	0.21 495	9.93 138	7	38	6 12.6 12.0
23	9.71 664	21	9.78 533	29	0.21 467	9.93 131	8	37	7 14.7 14.0
24	9.71 685	20	9.78 562	28	0.21 438	9.93 123	8	36	8 16.8 16.0
25	9.71 705	21	9.78 590	28	0.21 410	9.93 115	7	35	9 18.9 18.0
26	9.71 726	21	9.78 618	29	0.21 382	9.93 108	8	34	
27	9.71 747	20	9.78 647	28	0.21 353	9.93 100	8	33	
28	9.71 767	21	9.78 675	29	0.21 325	9.93 092	8	32	
29	9.71 788	21	9.78 704	28	0.21 296	9.93 084	7	31	
30	9.71 809	20	9.78 732	28	0.21 268	9.93 077	8	30	8 7
31	9.71 829	21	9.78 760	29	0.21 240	9.93 069	8	29	1 0.8 0.7
32	9.71 850	20	9.78 789	28	0.21 211	9.93 061	8	28	2 1.6 1.4
33	9.71 870	21	9.78 817	28	0.21 183	9.93 053	7	27	3 2.4 2.1
34	9.71 891	20	9.78 845	29	0.21 155	9.93 046	8	26	4 3.2 2.8
35	9.71 911	21	9.78 874	28	0.21 126	9.93 038	8	25	5 4.0 3.5
36	9.71 932	21	9.78 902	28	0.21 098	9.93 030	8	24	6 4.8 4.2
37	9.71 952	21	9.78 930	29	0.21 070	9.93 022	8	23	7 5.6 4.9
38	9.71 973	21	9.78 959	28	0.21 041	9.93 014	8	22	8 6.4 5.6
39	9.71 994	20	9.78 987	28	0.21 013	9.93 007	7	21	9 7.2 6.3
40	9.72 014	20	9.79 015	28	0.20 985	9.92 999	8	20	
41	9.72 034	21	9.79 043	29	0.20 957	9.92 991	8	19	
42	9.72 055	20	9.79 072	28	0.20 928	9.92 983	7	18	
43	9.72 075	21	9.79 100	28	0.20 900	9.92 976	8	17	
44	9.72 096	20	9.79 128	28	0.20 872	9.92 968	8	16	
45	9.72 116	21	9.79 156	29	0.20 844	9.92 960	8	15	
46	9.72 137	20	9.79 185	28	0.20 815	9.92 952	8	14	8 8 8
47	9.72 157	20	9.79 213	28	0.20 787	9.92 944	8	13	30 29 28
48	9.72 177	21	9.79 241	28	0.20 759	9.92 936	7	12	0 1.9 1.8
49	9.72 198	20	9.79 269	28	0.20 731	9.92 929	8	11	1 5.6 5.4
50	9.72 218	20	9.79 297	29	0.20 703	9.92 921	8	10	2 9.4 9.1
51	9.72 238	21	9.79 326	28	0.20 674	9.92 913	8	9	3 13.1 12.7
52	9.72 259	20	9.79 354	28	0.20 646	9.92 905	8	8	4 16.9 16.3
53	9.72 279	20	9.79 382	28	0.20 618	9.92 897	8	7	5 20.6 19.9
54	9.72 299	21	9.79 410	28	0.20 590	9.92 889	8	6	6 24.4 23.6
55	9.72 320	20	9.79 438	28	0.20 562	9.92 881	7	5	7 28.1 27.2
56	9.72 340	20	9.79 466	29	0.20 534	9.92 874	8	4	8 26.2
57	9.72 360	21	9.79 495	28	0.20 505	9.92 866	8	3	
58	9.72 381	20	9.79 523	28	0.20 477	9.92 858	8	2	
59	9.72 401	20	9.79 551	28	0.20 449	9.92 850	8	1	
60	9.72 421	20	9.79 579	28	0.20 421	9.92 842	8	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

'	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.72 421	20	9.79 579	28	0.20 421	9.92 842	8	60	
1	9.72 441	20	9.79 607	28	0.20 393	9.92 834	8	59	
2	9.72 461	21	9.79 635	28	0.20 365	9.92 826	8	58	
3	9.72 482	20	9.79 663	28	0.20 337	9.92 818	8	57	
4	9.72 502	20	9.79 691	28	0.20 309	9.92 810	8	56	29 28 27
5	9.72 522	20	9.79 719	28	0.20 281	9.92 803	7	55	1 2.9 2.8 2.7
6	9.72 542	20	9.79 747	29	0.20 253	9.92 795	8	54	2 5.8 5.6 5.4
7	9.72 562	20	9.79 776	28	0.20 224	9.92 787	8	53	3 8.7 8.4 8.1
8	9.72 582	20	9.79 804	28	0.20 196	9.92 779	8	52	4 11.6 11.2 10.8
9	9.72 602	20	9.79 832	28	0.20 168	9.92 771	8	51	5 14.5 14.0 13.5
10	9.72 622	21	9.79 860	28	0.20 140	9.92 763	8	50	6 17.4 16.8 16.2
11	9.72 643	20	9.79 888	28	0.20 112	9.92 755	8	49	7 20.3 19.6 18.9
12	9.72 663	20	9.79 916	28	0.20 084	9.92 747	8	48	8 23.2 22.4 21.6
13	9.72 683	20	9.79 944	28	0.20 056	9.92 739	8	47	9 26.1 25.2 24.3
14	9.72 703	20	9.79 972	28	0.20 028	9.92 731	8	46	
15	9.72 723	20	9.80 000	28	0.20 000	9.92 723	8	45	
16	9.72 743	20	9.80 028	28	0.19 972	9.92 715	8	44	
17	9.72 763	20	9.80 056	28	0.19 944	9.92 707	8	43	31 30 19
18	9.72 783	20	9.80 084	28	0.19 916	9.92 699	8	42	1 2.1 2.0 1.9
19	9.72 803	20	9.80 112	28	0.19 888	9.92 691	8	41	2 4.2 4.0 3.8
20	9.72 823	20	9.80 140	28	0.19 860	9.92 683	8	40	3 6.3 6.0 5.7
21	9.72 843	20	9.80 168	28	0.19 832	9.92 675	8	39	4 8.4 8.0 7.6
22	9.72 863	20	9.80 195	27	0.19 805	9.92 667	8	38	5 10.5 10.0 9.5
23	9.72 883	19	9.80 223	28	0.19 777	9.92 659	8	37	6 12.6 12.0 11.4
24	9.72 902	20	9.80 251	28	0.19 749	9.92 651	8	36	7 14.7 14.0 13.3
25	9.72 922	20	9.80 279	28	0.19 721	9.92 643	8	35	8 16.8 16.0 15.2
26	9.72 942	20	9.80 307	28	0.19 693	9.92 635	8	34	9 18.9 18.0 17.1
27	9.72 962	20	9.80 335	28	0.19 665	9.92 627	8	33	
28	9.72 982	20	9.80 363	28	0.19 637	9.92 619	8	32	
29	9.73 002	20	9.80 391	28	0.19 609	9.92 611	8	31	
30	9.73 022	19	9.80 419	28	0.19 581	9.92 603	8	30	9 8 7
31	9.73 041	20	9.80 447	27	0.19 553	9.92 595	8	29	1 0.9 0.8 0.7
32	9.73 061	20	9.80 474	28	0.19 526	9.92 587	8	28	2 1.8 1.6 1.4
33	9.73 081	20	9.80 502	28	0.19 498	9.92 579	8	27	3 2.7 2.4 2.1
34	9.73 101	20	9.80 530	28	0.19 470	9.92 571	8	26	4 3.6 3.2 2.8
35	9.73 121	19	9.80 558	28	0.19 442	9.92 563	8	25	5 4.5 4.0 3.5
36	9.73 140	20	9.80 586	28	0.19 414	9.92 555	8	24	6 5.4 4.8 4.2
37	9.73 160	20	9.80 614	28	0.19 386	9.92 546	9	23	7 6.3 5.6 4.9
38	9.73 180	20	9.80 642	27	0.19 358	9.92 538	8	22	8 7.2 6.4 5.6
39	9.73 200	19	9.80 669	28	0.19 331	9.92 530	8	21	9 8.1 7.2 6.3
40	9.73 219	20	9.80 697	28	0.19 303	9.92 522	8	20	
41	9.73 239	20	9.80 725	28	0.19 275	9.92 514	8	19	
42	9.73 259	19	9.80 753	28	0.19 247	9.92 506	8	18	
43	9.73 278	20	9.80 781	27	0.19 219	9.92 498	8	17	
44	9.73 298	20	9.80 808	28	0.19 192	9.92 490	8	16	
45	9.73 318	19	9.80 836	28	0.19 164	9.92 482	9	15	
46	9.73 337	20	9.80 864	28	0.19 136	9.92 473	8	14	
47	9.73 357	20	9.80 892	27	0.19 108	9.92 465	8	13	8 8 7
48	9.73 377	19	9.80 919	28	0.19 081	9.92 457	8	12	29 28 28
49	9.73 396	20	9.80 947	28	0.19 053	9.92 449	8	11	
50	9.73 416	19	9.80 975	28	0.19 025	9.92 441	8	10	0 1.8 1.8 2.0
51	9.73 435	20	9.81 003	27	0.18 997	9.92 433	8	9	1 5.4 5.2 6.0
52	9.73 455	19	9.81 030	28	0.18 970	9.92 425	8	8	2 9.1 8.8 11
53	9.73 474	20	9.81 058	28	0.18 942	9.92 416	9	7	3 12.7 12.2 11
54	9.73 494	19	9.81 086	27	0.18 914	9.92 408	8	6	4 16.3 15.8 11
55	9.73 513	20	9.81 113	28	0.18 887	9.92 400	8	5	5 19.9 19.2 21
56	9.73 533	19	9.81 141	28	0.18 859	9.92 392	8	4	6 23.6 22.8 21
57	9.73 552	20	9.81 169	27	0.18 831	9.92 384	8	3	7 27.2 26.2 21
58	9.73 572	19	9.81 196	28	0.18 804	9.92 376	9	2	
59	9.73 591	20	9.81 224	28	0.18 776	9.92 367	8	1	
60	9.73 611	20	9.81 252	28	0.18 748	9.92 359	8	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	'	P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.73 611	19	9.81 252	27	0.18 748	9.92 359	8	60	
1	9.73 630	19	9.81 279	28	0.18 721	9.92 351	8	59	
2	9.73 650	20	9.81 307	28	0.18 693	9.92 343	8	58	
3	9.73 669	20	9.81 335	27	0.18 665	9.92 335	9	57	28 27
4	9.73 689	19	9.81 362	28	0.18 638	9.92 326	8	56	
5	9.73 708	19	9.81 390	28	0.18 610	9.92 318	8	55	1 2.8 2.7
6	9.73 727	20	9.81 418	27	0.18 582	9.92 310	8	54	2 5.6 5.4
7	9.73 747	19	9.81 445	28	0.18 555	9.92 302	9	53	3 8.4 8.1
8	9.73 766	19	9.81 473	27	0.18 527	9.92 293	8	52	4 11.2 10.8
9	9.73 785	20	9.81 500	28	0.18 500	9.92 285	8	51	5 14.0 13.5
10	9.73 805	19	9.81 528	28	0.18 472	9.92 277	8	50	6 16.8 16.2
11	9.73 824	19	9.81 556	27	0.18 444	9.92 269	9	49	7 19.6 18.9
12	9.73 843	20	9.81 583	28	0.18 417	9.92 260	8	48	8 22.4 21.6
13	9.73 863	19	9.81 611	27	0.18 389	9.92 252	8	47	9 25.2 24.3
14	9.73 882	19	9.81 638	28	0.18 362	9.92 244	9	46	
15	9.73 901	20	9.81 666	27	0.18 334	9.92 235	8	45	
16	9.73 921	19	9.81 693	28	0.18 307	9.92 227	8	44	90 19 18
17	9.73 940	19	9.81 721	27	0.18 279	9.92 219	8	43	
18	9.73 959	19	9.81 748	28	0.18 252	9.92 211	9	42	1 2.0 1.9 1.8
19	9.73 978	19	9.81 776	27	0.18 224	9.92 202	8	41	2 4.0 3.8 3.6
20	9.73 997	20	9.81 803	28	0.18 197	9.92 194	8	40	3 6.0 5.7 5.4
21	9.74 017	19	9.81 831	27	0.18 169	9.92 186	9	39	4 8.0 7.6 7.2
22	9.74 036	19	9.81 858	28	0.18 142	9.92 177	8	38	5 10.0 9.5 9.0
23	9.74 055	19	9.81 886	27	0.18 114	9.92 169	8	37	6 12.0 11.4 10.8
24	9.74 074	19	9.81 913	28	0.18 087	9.92 161	9	36	7 14.0 13.3 12.6
25	9.74 093	20	9.81 941	27	0.18 059	9.92 152	8	35	8 16.0 15.2 14.4
26	9.74 113	19	9.81 968	28	0.18 032	9.92 144	8	34	9 18.0 17.1 16.2
27	9.74 132	19	9.81 996	27	0.18 004	9.92 136	9	33	
28	9.74 151	19	9.82 023	28	0.17 977	9.92 127	8	32	
29	9.74 170	19	9.82 051	27	0.17 949	9.92 119	8	31	
30	9.74 189	19	9.82 078	28	0.17 922	9.92 111	9	30	9 8
31	9.74 208	19	9.82 106	27	0.17 894	9.92 102	8	29	1 0.9 0.8
32	9.74 227	19	9.82 133	28	0.17 867	9.92 094	8	28	2 1.8 1.6
33	9.74 246	19	9.82 161	27	0.17 839	9.92 086	8	27	3 2.7 2.4
34	9.74 265	19	9.82 188	28	0.17 812	9.92 077	9	26	4 3.6 3.2
35	9.74 284	19	9.82 215	27	0.17 785	9.92 069	8	25	5 4.5 4.0
36	9.74 303	19	9.82 243	28	0.17 757	9.92 060	9	24	6 5.4 4.8
37	9.74 322	19	9.82 270	27	0.17 730	9.92 052	8	23	7 6.3 5.6
38	9.74 341	19	9.82 298	28	0.17 702	9.92 044	8	22	8 7.2 6.4
39	9.74 360	19	9.82 325	27	0.17 675	9.92 035	9	21	9 8.1 7.2
40	9.74 379	19	9.82 352	28	0.17 648	9.92 027	8	20	
41	9.74 398	19	9.82 380	27	0.17 620	9.92 018	9	19	
42	9.74 417	19	9.82 407	28	0.17 593	9.92 010	8	18	
43	9.74 436	19	9.82 435	27	0.17 565	9.92 002	8	17	
44	9.74 455	19	9.82 462	28	0.17 538	9.91 993	9	16	
45	9.74 474	19	9.82 489	27	0.17 511	9.91 985	8	15	
46	9.74 493	19	9.82 517	28	0.17 483	9.91 976	9	14	
47	9.74 512	19	9.82 544	27	0.17 456	9.91 968	8	13	9 9 8
48	9.74 531	18	9.82 571	28	0.17 429	9.91 959	9	12	28 27 27
49	9.74 549	19	9.82 599	27	0.17 401	9.91 951	8	11	
50	9.74 568	19	9.82 626	28	0.17 374	9.91 942	9	10	0 1.6 1.5 1.7
51	9.74 587	19	9.82 653	27	0.17 347	9.91 934	8	9	1 4.7 4.5 5.1
52	9.74 606	19	9.82 681	28	0.17 319	9.91 925	9	8	2 7.8 7.5 8.4
53	9.74 625	19	9.82 708	27	0.17 292	9.91 917	8	7	3 10.9 10.5 11.8
54	9.74 644	18	9.82 735	28	0.17 265	9.91 908	9	6	4 14.0 13.5 15.2
55	9.74 662	19	9.82 762	27	0.17 238	9.91 900	8	5	5 17.1 16.5 18.6
56	9.74 681	19	9.82 790	28	0.17 210	9.91 891	9	4	6 20.2 19.5 21.9
57	9.74 700	18	9.82 817	27	0.17 183	9.91 883	8	3	7 23.3 22.5 25.3
58	9.74 719	19	9.82 844	28	0.17 156	9.91 874	9	2	8 26.4 25.5 —
59	9.74 737	19	9.82 871	27	0.17 129	9.91 866	8	1	
60	9.74 756	19	9.82 899	28	0.17 101	9.91 857	9	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.74 756	19	9.82 899	27	0.17 101	9.91 857	8	60	
1	9.74 775	19	9.82 926	27	0.17 074	9.91 849	8	59	
2	9.74 794	18	9.82 953	27	0.17 047	9.91 840	8	58	
3	9.74 812	19	9.82 980	27	0.17 020	9.91 832	9	57	
4	9.74 831	19	9.83 008	27	0.16 992	9.91 823	8	56	28 27 26
5	9.74 850	18	9.83 035	27	0.16 965	9.91 815	8	55	1 2.8 2.7 2.6
6	9.74 868	19	9.83 062	27	0.16 938	9.91 806	8	54	2 5.6 5.4 5.2
7	9.74 887	19	9.83 089	27	0.16 911	9.91 798	8	53	3 8.4 8.1 7.8
8	9.74 906	18	9.83 117	27	0.16 883	9.91 789	8	52	4 11.2 10.8 10.4
9	9.74 924	19	9.83 144	27	0.16 856	9.91 781	8	51	5 14.0 13.5 13.0
10	9.74 943	18	9.83 171	27	0.16 829	9.91 772	9	50	6 16.8 16.2 15.6
11	9.74 961	19	9.83 198	27	0.16 802	9.91 763	8	49	7 19.6 18.9 18.2
12	9.74 980	19	9.83 225	27	0.16 775	9.91 755	8	48	8 22.4 21.6 20.8
13	9.74 999	18	9.83 252	28	0.16 748	9.91 746	8	47	9 25.2 24.3 23.4
14	9.75 017	19	9.83 280	27	0.16 720	9.91 738	9	46	
15	9.75 036	18	9.83 307	27	0.16 693	9.91 729	9	45	
16	9.75 054	19	9.83 334	27	0.16 666	9.91 720	8	44	
17	9.75 073	18	9.83 361	27	0.16 639	9.91 712	9	43	19 18
18	9.75 091	19	9.83 388	27	0.16 612	9.91 703	8	42	1 1.9 1.8
19	9.75 110	18	9.83 415	27	0.16 585	9.91 695	8	41	2 3.8 3.6
20	9.75 128	19	9.83 442	28	0.16 558	9.91 686	9	40	3 5.7 5.4
21	9.75 147	18	9.83 470	27	0.16 530	9.91 677	8	39	4 7.6 7.2
22	9.75 165	19	9.83 497	27	0.16 503	9.91 669	8	38	5 9.5 9.0
23	9.75 184	18	9.83 524	27	0.16 476	9.91 660	9	37	6 11.4 10.8
24	9.75 202	19	9.83 551	27	0.16 449	9.91 651	8	36	7 13.3 12.6
25	9.75 221	18	9.83 578	27	0.16 422	9.91 643	8	35	8 15.2 14.4
26	9.75 239	19	9.83 605	27	0.16 395	9.91 634	9	34	9 17.1 16.2
27	9.75 258	18	9.83 632	27	0.16 368	9.91 625	8	33	
28	9.75 276	19	9.83 659	27	0.16 341	9.91 617	8	32	
29	9.75 294	18	9.83 686	27	0.16 314	9.91 608	9	31	
30	9.75 313	18	9.83 713	27	0.16 287	9.91 599	8	30	9 8
31	9.75 331	19	9.83 740	28	0.16 260	9.91 591	9	29	1 0.9 0.8
32	9.75 350	18	9.83 768	27	0.16 232	9.91 582	9	28	2 1.8 1.6
33	9.75 368	18	9.83 795	27	0.16 205	9.91 573	8	27	3 2.7 2.4
34	9.75 386	19	9.83 822	27	0.16 178	9.91 565	8	26	4 3.6 3.2
35	9.75 405	18	9.83 849	27	0.16 151	9.91 556	9	25	5 4.5 4.0
36	9.75 423	18	9.83 876	27	0.16 124	9.91 547	9	24	6 5.4 4.8
37	9.75 441	18	9.83 903	27	0.16 097	9.91 538	8	23	7 6.3 5.6
38	9.75 459	19	9.83 930	27	0.16 070	9.91 530	8	22	8 7.2 6.4
39	9.75 478	18	9.83 957	27	0.16 043	9.91 521	9	21	9 8.1 7.2
40	9.75 496	18	9.83 984	27	0.16 016	9.91 512	8	20	
41	9.75 514	19	9.84 011	27	0.15 989	9.91 504	9	19	
42	9.75 533	18	9.84 038	27	0.15 962	9.91 495	9	18	
43	9.75 551	18	9.84 065	27	0.15 935	9.91 486	9	17	
44	9.75 569	18	9.84 092	27	0.15 908	9.91 477	8	16	
45	9.75 587	18	9.84 119	27	0.15 881	9.91 469	8	15	
46	9.75 605	19	9.84 146	27	0.15 854	9.91 460	9	14	
47	9.75 624	18	9.84 173	27	0.15 827	9.91 451	9	13	9 8 8
48	9.75 642	18	9.84 200	27	0.15 800	9.91 442	9	12	28 28 27
49	9.75 660	18	9.84 227	27	0.15 773	9.91 433	8	11	1 1.6 1.8 1.7
50	9.75 678	18	9.84 254	26	0.15 746	9.91 425	9	10	2 4.7 5.2 5.1
51	9.75 696	18	9.84 280	27	0.15 720	9.91 416	9	9	3 7.8 8.8 8.4
52	9.75 714	19	9.84 307	27	0.15 693	9.91 407	9	8	4 10.9 12.2 11
53	9.75 733	18	9.84 334	27	0.15 666	9.91 398	9	7	5 14.0 15.8 15
54	9.75 751	18	9.84 361	27	0.15 639	9.91 389	8	6	6 17.1 19.2 18
55	9.75 769	18	9.84 388	27	0.15 612	9.91 381	8	5	7 20.2 22.8 21
56	9.75 787	18	9.84 415	27	0.15 585	9.91 372	9	4	8 23.3 26.2 25
57	9.75 805	18	9.84 442	27	0.15 558	9.91 363	9	3	9 26.4 — —
58	9.75 823	18	9.84 469	27	0.15 531	9.91 354	9	2	
59	9.75 841	18	9.84 496	27	0.15 504	9.91 345	9	1	
60	9.75 859		9.84 523	27	0.15 477	9.91 336	9	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.75 859	18	9.84 523	27	0.15 477	9.91 336	8	60	
1	9.75 877	18	9.84 530	26	0.15 450	9.91 328	9	59	27 26
2	9.75 895	18	9.84 576	27	0.15 424	9.91 319	9	58	1 2.7 2.6
3	9.75 913	18	9.84 603	27	0.15 397	9.91 310	9	57	2 5.4 5.2
4	9.75 931	18	9.84 630	27	0.15 370	9.91 301	9	56	3 8.1 7.8
5	9.75 949	18	9.84 657	27	0.15 343	9.91 292	9	55	4 10.8 10.4
6	9.75 967	18	9.84 684	27	0.15 316	9.91 283	9	54	5 13.5 13.0
7	9.75 985	18	9.84 711	27	0.15 289	9.91 274	9	53	6 16.2 15.6
8	9.76 003	18	9.84 738	27	0.15 262	9.91 266	8	52	7 18.9 18.2
9	9.76 021	18	9.84 764	26	0.15 236	9.91 257	9	51	8 21.6 20.8
10	9.76 039	18	9.84 791	27	0.15 209	9.91 248	9	50	9 24.3 23.4
11	9.76 057	18	9.84 818	27	0.15 182	9.91 239	9	49	
12	9.76 075	18	9.84 845	27	0.15 155	9.91 230	9	48	18 17
13	9.76 093	18	9.84 872	27	0.15 128	9.91 221	9	47	1 1.8 1.7
14	9.76 111	18	9.84 899	27	0.15 101	9.91 212	9	46	2 3.6 3.4
15	9.76 129	17	9.84 925	26	0.15 075	9.91 203	9	45	3 5.4 5.1
16	9.76 146	18	9.84 952	27	0.15 048	9.91 194	9	44	4 7.2 6.8
17	9.76 164	18	9.84 979	27	0.15 021	9.91 185	9	43	5 9.0 8.5
18	9.76 182	18	9.85 006	27	0.14 994	9.91 176	9	42	6 10.8 10.2
19	9.76 200	18	9.85 033	26	0.14 967	9.91 167	9	41	7 12.6 11.9
20	9.76 218	18	9.85 059	27	0.14 941	9.91 158	9	40	8 14.4 13.6
21	9.76 236	17	9.85 086	27	0.14 914	9.91 149	9	39	9 16.2 15.3
22	9.76 253	18	9.85 113	27	0.14 887	9.91 141	8	38	
23	9.76 271	18	9.85 140	26	0.14 860	9.91 132	9	37	10 9 8
24	9.76 289	18	9.85 166	27	0.14 834	9.91 123	9	36	1 1.0 0.9 0.8
25	9.76 307	17	9.85 193	27	0.14 807	9.91 114	9	35	2 2.0 1.8 1.6
26	9.76 324	18	9.85 220	27	0.14 780	9.91 105	9	34	3 3.0 2.7 2.4
27	9.76 342	18	9.85 247	27	0.14 753	9.91 096	9	33	4 4.0 3.6 3.2
28	9.76 360	18	9.85 273	26	0.14 727	9.91 087	9	32	5 5.0 4.5 4.0
29	9.76 378	17	9.85 300	27	0.14 700	9.91 078	9	31	6 6.0 5.4 4.8
30	9.76 395	18	9.85 327	27	0.14 673	9.91 069	9	30	7 7.0 6.3 5.6
31	9.76 413	18	9.85 354	26	0.14 646	9.91 060	9	29	8 8.0 7.2 6.4
32	9.76 431	17	9.85 380	27	0.14 620	9.91 051	9	28	9 9.0 8.1 7.2
33	9.76 448	18	9.85 407	27	0.14 593	9.91 042	9	27	
34	9.76 466	18	9.85 434	26	0.14 566	9.91 033	9	26	
35	9.76 484	17	9.85 460	27	0.14 540	9.91 023	10	25	10 10
36	9.76 501	18	9.85 487	27	0.14 513	9.91 014	9	24	27 26
37	9.76 519	18	9.85 514	26	0.14 486	9.91 005	9	23	0 1.4 1.3
38	9.76 537	17	9.85 540	27	0.14 460	9.90 996	9	22	1 4.0 3.9
39	9.76 554	18	9.85 567	27	0.14 433	9.90 987	9	21	2 6.8 6.5
40	9.76 572	18	9.85 594	26	0.14 406	9.90 978	9	20	3 9.4 9.1
41	9.76 590	17	9.85 620	27	0.14 380	9.90 969	9	19	4 12.2 11.7
42	9.76 607	18	9.85 647	27	0.14 353	9.90 960	9	18	5 14.8 14.3
43	9.76 623	17	9.85 674	26	0.14 326	9.90 951	9	17	6 17.6 16.9
44	9.76 642	18	9.85 700	27	0.14 300	9.90 942	9	16	7 20.2 19.5
45	9.76 660	17	9.85 727	27	0.14 273	9.90 933	9	15	8 23.0 22.1
46	9.76 677	18	9.85 754	26	0.14 246	9.90 924	9	14	9 25.6 24.7
47	9.76 693	17	9.85 780	27	0.14 220	9.90 915	9	13	
48	9.76 712	18	9.85 807	27	0.14 193	9.90 906	9	12	
49	9.76 730	17	9.85 834	26	0.14 166	9.90 896	10	11	9 9
50	9.76 747	18	9.85 860	27	0.14 140	9.90 887	9	10	27 26
51	9.76 765	17	9.85 887	26	0.14 113	9.90 878	9	9	0 1.5 1.4
	9.76 782	18	9.85 913	27	0.14 087	9.90 869	9	8	1 4.5 4.3
	9.76 800	17	9.85 940	27	0.14 060	9.90 860	9	7	2 7.5 7.2
	9.76 817	18	9.85 967	26	0.14 033	9.90 851	9	6	3 10.5 10.1
	9.76 835	17	9.85 993	27	0.14 007	9.90 842	9	5	4 13.5 13.0
	9.76 852	18	9.86 020	26	0.13 980	9.90 832	10	4	5 16.5 15.9
	9.76 870	17	9.86 046	27	0.13 954	9.90 823	9	3	6 19.5 18.8
	9.76 887	18	9.86 073	27	0.13 927	9.90 814	9	2	7 22.5 21.7
	9.76 904	17	9.86 100	26	0.13 900	9.90 805	9	1	8 25.5 24.6
52	9.76 922	18	9.86 126		0.13 874	9.90 796	9	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.76 922	17	9.86 126	27	0.13 874	9.90 796	9	60	
1	9.76 939	18	9.86 153	26	0.13 847	9.90 787	10	59	
2	9.76 957	17	9.86 179	27	0.13 821	9.90 777	9	58	
3	9.76 974	17	9.86 206	26	0.13 794	9.90 768	9	57	
4	9.76 991	18	9.86 232	27	0.13 768	9.90 759	9	56	27 28
5	9.77 009	17	9.86 259	26	0.13 741	9.90 750	9	55	1 2.7 2.6
6	9.77 026	17	9.86 285	26	0.13 715	9.90 741	9	54	2 5.4 5.2
7	9.77 043	17	9.86 312	27	0.13 688	9.90 731	10	53	3 8.1 7.8
8	9.77 061	18	9.86 338	26	0.13 662	9.90 722	9	52	4 10.8 10.4
9	9.77 078	17	9.86 365	27	0.13 635	9.90 713	9	51	5 13.5 13.0
10	9.77 095	17	9.86 392	26	0.13 608	9.90 704	9	50	6 16.2 15.6
11	9.77 112	18	9.86 418	27	0.13 582	9.90 694	10	49	7 18.9 18.2
12	9.77 130	17	9.86 445	26	0.13 555	9.90 685	9	48	8 21.6 20.8
13	9.77 147	17	9.86 471	27	0.13 529	9.90 676	9	47	9 24.3 23.4
14	9.77 164	17	9.86 498	26	0.13 502	9.90 667	9	46	
15	9.77 181	18	9.86 524	27	0.13 476	9.90 657	10	45	
16	9.77 199	17	9.86 551	26	0.13 449	9.90 648	9	44	18 17 16
17	9.77 216	17	9.86 577	26	0.13 423	9.90 639	9	43	
18	9.77 233	17	9.86 603	27	0.13 397	9.90 630	10	42	1 1.8 1.7 1.6
19	9.77 250	18	9.86 630	26	0.13 370	9.90 620	9	41	2 3.6 3.4 3.2
20	9.77 268	17	9.86 656	27	0.13 344	9.90 611	9	40	3 5.4 5.1 4.8
21	9.77 285	17	9.86 683	26	0.13 317	9.90 602	9	39	4 7.2 6.8 6.4
22	9.77 302	17	9.86 709	27	0.13 291	9.90 592	10	38	5 9.0 8.5 8.0
23	9.77 319	17	9.86 736	26	0.13 264	9.90 583	9	37	6 10.8 10.2 9.6
24	9.77 336	17	9.86 762	27	0.13 238	9.90 574	9	36	7 12.6 11.9 11.2
25	9.77 353	17	9.86 789	26	0.13 211	9.90 565	10	35	8 14.4 13.6 12.8
26	9.77 370	17	9.86 815	27	0.13 185	9.90 555	9	34	9 16.2 15.3 14.4
27	9.77 387	18	9.86 842	26	0.13 158	9.90 546	9	33	
28	9.77 405	17	9.86 868	26	0.13 132	9.90 537	9	32	
29	9.77 422	17	9.86 894	27	0.13 106	9.90 527	10	31	
30	9.77 439	17	9.86 921	26	0.13 079	9.90 518	9	30	10 9
31	9.77 456	17	9.86 947	27	0.13 053	9.90 509	9	29	1 1.0 0.9
32	9.77 473	17	9.86 974	26	0.13 026	9.90 499	10	28	2 2.0 1.8
33	9.77 490	17	9.87 000	27	0.13 000	9.90 490	9	27	3 3.0 2.7
34	9.77 507	17	9.87 027	26	0.12 973	9.90 480	10	26	4 4.0 3.6
35	9.77 524	17	9.87 053	26	0.12 947	9.90 471	9	25	5 5.0 4.5
36	9.77 541	17	9.87 079	27	0.12 921	9.90 462	9	24	6 6.0 5.4
37	9.77 558	17	9.87 106	26	0.12 894	9.90 452	10	23	7 7.0 6.3
38	9.77 575	17	9.87 132	26	0.12 868	9.90 443	9	22	8 8.0 7.2
39	9.77 592	17	9.87 158	27	0.12 842	9.90 434	9	21	9 9.0 8.1
40	9.77 609	17	9.87 185	26	0.12 815	9.90 424	10	20	
41	9.77 626	17	9.87 211	27	0.12 789	9.90 415	9	19	
42	9.77 643	17	9.87 238	26	0.12 762	9.90 405	10	18	
43	9.77 660	17	9.87 264	26	0.12 736	9.90 396	9	17	
44	9.77 677	17	9.87 290	27	0.12 710	9.90 386	10	16	
45	9.77 694	17	9.87 317	26	0.12 683	9.90 377	9	15	
46	9.77 711	17	9.87 343	26	0.12 657	9.90 368	9	14	9 9
47	9.77 728	16	9.87 369	27	0.12 631	9.90 358	9	13	27 28
48	9.77 744	17	9.87 396	26	0.12 604	9.90 349	10	12	
49	9.77 761	17	9.87 422	26	0.12 578	9.90 339	9	11	0 1.5 1.4
50	9.77 778	17	9.87 448	27	0.12 552	9.90 330	10	10	1 4.5 4.3
51	9.77 795	17	9.87 475	26	0.12 525	9.90 320	9	9	2 7.5 7.2
52	9.77 812	17	9.87 501	26	0.12 499	9.90 311	9	8	3 10.5 10.1
53	9.77 829	17	9.87 527	27	0.12 473	9.90 301	10	7	4 13.5 13.0
54	9.77 846	16	9.87 554	26	0.12 446	9.90 292	9	6	5 16.5 15.9
55	9.77 862	17	9.87 580	26	0.12 420	9.90 282	10	5	6 19.5 18.8
56	9.77 879	17	9.87 606	27	0.12 394	9.90 273	9	4	7 22.5 21.7
57	9.77 896	17	9.87 633	26	0.12 367	9.90 263	10	3	8 25.5 24.6
58	9.77 913	17	9.87 659	26	0.12 341	9.90 254	9	2	
59	9.77 930	16	9.87 685	26	0.12 315	9.90 244	10	1	
60	9.77 946		9.87 711		0.12 289	9.90 235	9	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.77 946	17	9.87 711	27	0.12 289	9.90 235	10	60	
1	9.77 963	17	9.87 738	26	0.12 262	9.90 225	9	59	
2	9.77 980	17	9.87 764	26	0.12 236	9.90 216	10	58	
3	9.77 997	16	9.87 790	27	0.12 210	9.90 206	9	57	
4	9.78 013	17	9.87 817	26	0.12 183	9.90 197	10	56	27 26
5	9.78 030	17	9.87 843	26	0.12 157	9.90 187	9	55	1 2.7 2.6
6	9.78 047	17	9.87 869	26	0.12 131	9.90 178	9	54	2 5.4 5.2
7	9.78 063	16	9.87 895	26	0.12 105	9.90 168	10	53	3 8.1 7.8
8	9.78 080	17	9.87 922	27	0.12 078	9.90 159	9	52	4 10.8 10.4
9	9.78 097	17	9.87 948	26	0.12 052	9.90 149	10	51	5 13.5 13.0
10	9.78 113	16	9.87 974	26	0.12 026	9.90 139	10	50	6 16.2 15.6
11	9.78 130	17	9.88 000	27	0.12 000	9.90 130	9	49	7 18.9 18.2
12	9.78 147	17	9.88 027	26	0.11 973	9.90 120	10	48	8 21.6 20.8
13	9.78 163	16	9.88 053	26	0.11 947	9.90 111	9	47	9 24.3 23.4
14	9.78 180	17	9.88 079	26	0.11 921	9.90 101	10	46	
15	9.78 197	17	9.88 105	26	0.11 895	9.90 091	9	45	
16	9.78 213	16	9.88 131	27	0.11 869	9.90 082	10	44	
17	9.78 230	17	9.88 158	26	0.11 842	9.90 072	9	43	17 16
18	9.78 246	16	9.88 184	26	0.11 816	9.90 063	10	42	1 1.7 1.6
19	9.78 263	17	9.88 210	26	0.11 790	9.90 053	10	41	2 3.4 3.2
20	9.78 280	16	9.88 236	26	0.11 764	9.90 043	9	40	3 5.1 4.8
21	9.78 296	17	9.88 262	27	0.11 738	9.90 034	10	39	4 6.8 6.4
22	9.78 313	16	9.88 289	26	0.11 711	9.90 024	9	38	5 8.5 8.0
23	9.78 329	17	9.88 315	26	0.11 685	9.90 014	10	37	6 10.2 9.6
24	9.78 346	16	9.88 341	26	0.11 659	9.90 005	9	36	7 11.9 11.2
25	9.78 362	17	9.88 367	26	0.11 633	9.89 995	10	35	8 13.6 12.8
26	9.78 379	16	9.88 393	27	0.11 607	9.89 985	9	34	9 15.3 14.4
27	9.78 395	17	9.88 420	26	0.11 580	9.89 976	10	33	
28	9.78 412	16	9.88 446	26	0.11 554	9.89 966	9	32	
29	9.78 428	17	9.88 472	26	0.11 528	9.89 956	10	31	
30	9.78 445	16	9.88 498	26	0.11 502	9.89 947	9	30	10 9
31	9.78 461	17	9.88 524	26	0.11 476	9.89 937	10	29	1 1.0 0.9
32	9.78 478	16	9.88 550	27	0.11 450	9.89 927	9	28	2 2.0 1.8
33	9.78 494	16	9.88 577	26	0.11 423	9.89 918	10	27	3 3.0 2.7
34	9.78 510	17	9.88 603	26	0.11 397	9.89 908	9	26	4 4.0 3.6
35	9.78 527	16	9.88 629	26	0.11 371	9.89 898	10	25	5 5.0 4.5
36	9.78 543	17	9.88 655	26	0.11 345	9.89 888	9	24	6 6.0 5.4
37	9.78 560	16	9.88 681	26	0.11 319	9.89 879	10	23	7 7.0 6.3
38	9.78 576	17	9.88 707	26	0.11 293	9.89 869	9	22	8 8.0 7.2
39	9.78 592	16	9.88 733	26	0.11 267	9.89 859	10	21	9 9.0 8.1
40	9.78 609	17	9.88 759	27	0.11 241	9.89 849	9	20	
41	9.78 625	16	9.88 786	26	0.11 214	9.89 840	10	19	
42	9.78 642	17	9.88 812	26	0.11 188	9.89 830	9	18	
43	9.78 658	16	9.88 838	26	0.11 162	9.89 820	10	17	
44	9.78 674	17	9.88 864	26	0.11 136	9.89 810	9	16	
45	9.78 691	16	9.88 890	26	0.11 110	9.89 801	10	15	
46	9.78 707	17	9.88 916	26	0.11 084	9.89 791	9	14	10 10
47	9.78 723	16	9.88 942	26	0.11 058	9.89 781	10	13	27 26
48	9.78 739	17	9.88 968	26	0.11 032	9.89 771	9	12	0 1.4 1.3
49	9.78 756	16	9.88 994	26	0.11 006	9.89 761	10	11	1 4.0 3.9
50	9.78 772	17	9.89 020	26	0.10 980	9.89 752	9	10	2 6.8 6.5
51	9.78 788	16	9.89 046	27	0.10 954	9.89 742	10	9	3 9.4 9.1
	9.78 805	17	9.89 073	26	0.10 927	9.89 732	9	8	4 12.2 11.7
	9.78 821	16	9.89 099	26	0.10 901	9.89 722	10	7	5 14.8 14.3
	9.78 837	17	9.89 125	26	0.10 875	9.89 712	9	6	6 17.6 16.9
	9.78 853	16	9.89 151	26	0.10 849	9.89 702	10	5	7 20.2 19.5
	9.78 869	17	9.89 177	26	0.10 823	9.89 693	9	4	8 23.0 22.1
	9.78 886	16	9.89 203	26	0.10 797	9.89 683	10	3	9 25.6 24.7
	9.78 902	17	9.89 229	26	0.10 771	9.89 673	9	2	
	9.78 918	16	9.89 255	26	0.10 745	9.89 663	10	1	
	9.78 934	17	9.89 281	26	0.10 719	9.89 653	9	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.78 934	16	9.89 281	26	0.10 719	9.89 653	10	60	
1	9.78 950	17	9.89 307	26	0.10 693	9.89 643	10	59	
2	9.78 967	16	9.89 333	26	0.10 667	9.89 633	10	58	
3	9.78 983	16	9.89 359	26	0.10 641	9.89 624	9	57	26 25
4	9.78 999	16	9.89 385	26	0.10 615	9.89 614	10	56	1 2.6 2.5
5	9.79 015	16	9.89 411	26	0.10 589	9.89 604	10	55	2 5.2 5.0
6	9.79 031	16	9.89 437	26	0.10 563	9.89 594	10	54	3 7.8 7.5
7	9.79 047	16	9.89 463	26	0.10 537	9.89 584	10	53	4 10.4 10.0
8	9.79 063	16	9.89 489	26	0.10 511	9.89 574	10	52	5 13.0 12.5
9	9.79 079	16	9.89 515	26	0.10 485	9.89 564	10	51	6 15.6 15.0
10	9.79 095	16	9.89 541	26	0.10 459	9.89 554	10	50	7 18.2 17.5
11	9.79 111	17	9.89 567	26	0.10 433	9.89 544	10	49	8 20.8 20.0
12	9.79 128	16	9.89 593	26	0.10 407	9.89 534	10	48	9 23.4 22.5
13	9.79 144	16	9.89 619	26	0.10 381	9.89 524	10	47	
14	9.79 160	16	9.89 645	26	0.10 355	9.89 514	10	46	
15	9.79 176	16	9.89 671	26	0.10 329	9.89 504	9	45	
16	9.79 192	16	9.89 697	26	0.10 303	9.89 495	10	44	17 16 15
17	9.79 208	16	9.89 723	26	0.10 277	9.89 485	10	43	1 1.7 1.6 1.5
18	9.79 224	16	9.89 749	26	0.10 251	9.89 475	10	42	2 3.4 3.2 3.0
19	9.79 240	16	9.89 775	26	0.10 225	9.89 465	10	41	3 5.1 4.8 4.5
20	9.79 256	16	9.89 801	26	0.10 199	9.89 455	10	40	4 6.8 6.4 6.0
21	9.79 272	16	9.89 827	26	0.10 173	9.89 445	10	39	5 8.5 8.0 7.5
22	9.79 288	16	9.89 853	26	0.10 147	9.89 435	10	38	6 10.2 9.6 9.0
23	9.79 304	15	9.89 879	26	0.10 121	9.89 425	10	37	7 11.9 11.2 10.5
24	9.79 319	16	9.89 905	26	0.10 095	9.89 415	10	36	8 13.6 12.8 12.0
25	9.79 335	16	9.89 931	26	0.10 069	9.89 405	10	35	9 15.3 14.4 13.5
26	9.79 351	16	9.89 957	26	0.10 043	9.89 395	10	34	
27	9.79 367	16	9.89 983	26	0.10 017	9.89 385	10	33	
28	9.79 383	16	9.90 009	26	0.09 991	9.89 375	10	32	
29	9.79 399	16	9.90 035	26	0.09 965	9.89 364	10	31	11 10 9
30	9.79 415	16	9.90 061	25	0.09 939	9.89 354	10	30	1 1.1 1.0 0.9
31	9.79 431	16	9.90 086	26	0.09 914	9.89 344	10	29	2 2.2 2.0 1.8
32	9.79 447	16	9.90 112	26	0.09 888	9.89 334	10	28	3 3.3 3.0 2.7
33	9.79 463	15	9.90 138	26	0.09 862	9.89 324	10	27	4 4.4 4.0 3.6
34	9.79 478	16	9.90 164	26	0.09 836	9.89 314	10	26	5 5.5 5.0 4.5
35	9.79 494	16	9.90 190	26	0.09 810	9.89 304	10	25	6 6.6 6.0 5.4
36	9.79 510	16	9.90 216	26	0.09 784	9.89 294	10	24	7 7.7 7.0 6.3
37	9.79 526	16	9.90 242	26	0.09 758	9.89 284	10	23	8 8.8 8.0 7.2
38	9.79 542	16	9.90 268	26	0.09 732	9.89 274	10	22	9 9.9 9.0 8.1
39	9.79 558	15	9.90 294	26	0.09 706	9.89 264	10	21	
40	9.79 573	16	9.90 320	26	0.09 680	9.89 254	10	20	
41	9.79 589	16	9.90 346	25	0.09 654	9.89 244	10	19	
42	9.79 605	16	9.90 371	26	0.09 629	9.89 233	11	18	
43	9.79 621	15	9.90 397	26	0.09 603	9.89 223	10	17	
44	9.79 636	16	9.90 423	26	0.09 577	9.89 213	10	16	
45	9.79 652	16	9.90 449	26	0.09 551	9.89 203	10	15	
46	9.79 668	16	9.90 475	26	0.09 525	9.89 193	10	14	10 10 9
47	9.79 684	15	9.90 501	26	0.09 499	9.89 183	10	13	26 25 26
48	9.79 699	16	9.90 527	26	0.09 473	9.89 173	10	12	0 1.3 1.2 1.4
49	9.79 715	16	9.90 553	25	0.09 447	9.89 162	11	11	1 3.9 3.8 4.3
50	9.79 731	15	9.90 578	26	0.09 422	9.89 152	10	10	2 6.5 6.2 7.2
51	9.79 746	16	9.90 604	26	0.09 396	9.89 142	10	9	3 9.1 8.8 10.1
52	9.79 762	16	9.90 630	26	0.09 370	9.89 132	10	8	4 11.7 11.2 13
53	9.79 778	15	9.90 656	26	0.09 344	9.89 122	10	7	5 14.3 13.8 15
54	9.79 793	16	9.90 682	26	0.09 318	9.89 112	11	6	6 16.9 16.2 18
55	9.79 809	16	9.90 708	26	0.09 292	9.89 101	10	5	7 19.5 18.8 21
56	9.79 825	15	9.90 734	25	0.09 266	9.89 091	10	4	8 22.1 21.2 24
57	9.79 840	16	9.90 759	26	0.09 241	9.89 081	10	3	9 24.7 23.8 -
58	9.79 856	16	9.90 785	26	0.09 215	9.89 071	10	2	
59	9.79 872	15	9.90 811	26	0.09 189	9.89 060	11	1	
60	9.79 887	15	9.90 837	26	0.09 163	9.89 050	10	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.79 887	16	9.90 837	26	0.09 163	9.89 050	10	60	
1	9.79 903	15	9.90 863	26	0.09 137	9.89 040	10	59	
2	9.79 918	16	9.90 889	25	0.09 111	9.89 030	10	58	
3	9.79 934	16	9.90 914	26	0.09 086	9.89 020	11	57	26 25
4	9.79 950	15	9.90 940	26	0.09 060	9.89 009	10	56	
5	9.79 965	16	9.90 966	26	0.09 034	9.88 999	10	55	1 2.6 2.5
6	9.79 981	16	9.90 992	26	0.09 008	9.88 989	10	54	2 5.2 5.0
7	9.79 996	15	9.91 018	26	0.08 982	9.88 978	11	53	3 7.8 7.5
8	9.80 012	16	9.91 043	25	0.08 957	9.88 968	10	52	4 10.4 10.0
9	9.80 027	15	9.91 069	26	0.08 931	9.88 958	10	51	5 13.0 12.5
10	9.80 043	16	9.91 095	26	0.08 905	9.88 948	10	50	6 15.6 15.0
11	9.80 058	15	9.91 121	26	0.08 879	9.88 937	11	49	7 18.2 17.5
12	9.80 074	16	9.91 147	26	0.08 853	9.88 927	10	48	8 20.8 20.0
13	9.80 089	15	9.91 172	25	0.08 828	9.88 917	10	47	9 23.4 22.5
14	9.80 105	16	9.91 198	26	0.08 802	9.88 906	11	46	
15	9.80 120	15	9.91 224	26	0.08 776	9.88 896	10	45	
16	9.80 136	16	9.91 250	26	0.08 750	9.88 886	10	44	16 15
17	9.80 151	15	9.91 276	26	0.08 724	9.88 875	11	43	1 1.6 1.5
18	9.80 166	15	9.91 301	25	0.08 699	9.88 865	10	42	2 3.2 3.0
19	9.80 182	16	9.91 327	26	0.08 673	9.88 855	10	41	3 4.8 4.5
20	9.80 197	15	9.91 353	26	0.08 647	9.88 844	11	40	4 6.4 6.0
21	9.80 213	16	9.91 379	26	0.08 621	9.88 834	10	39	5 8.0 7.5
22	9.80 228	15	9.91 404	25	0.08 596	9.88 824	10	38	6 9.6 9.0
23	9.80 244	16	9.91 430	26	0.08 570	9.88 813	11	37	7 11.2 10.5
24	9.80 259	15	9.91 456	26	0.08 544	9.88 803	10	36	8 12.8 12.0
25	9.80 274	16	9.91 482	25	0.08 518	9.88 793	10	35	9 14.4 13.5
26	9.80 290	16	9.91 507	25	0.08 493	9.88 782	11	34	
27	9.80 305	15	9.91 533	26	0.08 467	9.88 772	10	33	
28	9.80 320	15	9.91 559	26	0.08 441	9.88 761	11	32	11 10
29	9.80 336	16	9.91 585	25	0.08 415	9.88 751	10	31	1 1.1 1.0
30	9.80 351	15	9.91 610	26	0.08 390	9.88 741	10	30	2 2.2 2.0
31	9.80 366	16	9.91 636	26	0.08 364	9.88 730	11	29	3 3.3 3.0
32	9.80 382	15	9.91 662	26	0.08 338	9.88 720	10	28	4 4.4 4.0
33	9.80 397	16	9.91 688	26	0.08 312	9.88 709	11	27	5 5.5 5.0
34	9.80 412	15	9.91 713	25	0.08 287	9.88 699	10	26	6 6.6 6.0
35	9.80 428	16	9.91 739	26	0.08 261	9.88 688	11	25	7 7.7 7.0
36	9.80 443	15	9.91 765	26	0.08 235	9.88 678	10	24	8 8.8 8.0
37	9.80 458	15	9.91 791	26	0.08 209	9.88 668	10	23	9 9.9 9.0
38	9.80 473	16	9.91 816	25	0.08 184	9.88 657	11	22	
39	9.80 489	16	9.91 842	26	0.08 158	9.88 647	10	21	
40	9.80 504	15	9.91 868	26	0.08 132	9.88 636	11	20	
41	9.80 519	15	9.91 893	25	0.08 107	9.88 626	10	19	
42	9.80 534	16	9.91 919	26	0.08 081	9.88 615	11	18	
43	9.80 550	16	9.91 945	26	0.08 055	9.88 605	10	17	
44	9.80 565	15	9.91 971	26	0.08 029	9.88 594	11	16	
45	9.80 580	15	9.91 996	25	0.08 004	9.88 584	10	15	11 11
46	9.80 595	15	9.92 022	26	0.07 978	9.88 573	11	14	26 25
47	9.80 610	15	9.92 048	26	0.07 952	9.88 563	10	13	
48	9.80 625	15	9.92 073	25	0.07 927	9.88 552	11	12	0 1.2 1.1
49	9.80 641	16	9.92 099	26	0.07 901	9.88 542	10	11	1 3.5 3.4
50	9.80 656	15	9.92 125	26	0.07 875	9.88 531	11	10	2 5.9 5.7
51	9.80 671	15	9.92 150	25	0.07 850	9.88 521	10	9	3 8.3 8.0
	9.80 686	15	9.92 176	26	0.07 824	9.88 510	11	8	4 10.6 10.2
	9.80 701	15	9.92 202	26	0.07 798	9.88 499	10	7	5 13.0 12.5
	9.80 716	15	9.92 227	25	0.07 773	9.88 489	11	6	6 15.4 14.8
	9.80 731	15	9.92 253	26	0.07 747	9.88 478	10	5	7 17.7 17.0
	9.80 746	16	9.92 279	26	0.07 721	9.88 468	11	4	8 20.1 19.3
	9.80 762	16	9.92 304	25	0.07 696	9.88 457	10	3	9 22.5 21.6
	9.80 777	15	9.92 330	26	0.07 670	9.88 447	11	2	10 24.8 23.9
	9.80 792	15	9.92 356	26	0.07 644	9.88 436	10	1	
	9.80 807	15	9.92 381	25	0.07 619	9.88 425	11	0	
	Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.80 807		9.92 381	26	0.07 619	9.88 425	10	60	
1	9.80 822	15	9.92 407	26	0.07 593	9.88 415	11	59	
2	9.80 837	15	9.92 433	25	0.07 567	9.88 404	10	58	
3	9.80 852	15	9.92 458	26	0.07 542	9.88 394	11	57	26 25
4	9.80 867	15	9.92 484	26	0.07 516	9.88 383	11	56	1 2.6 2.5
5	9.80 882	15	9.92 510	25	0.07 490	9.88 372	10	55	2 5.2 5.0
6	9.80 897	15	9.92 535	26	0.07 465	9.88 362	11	54	3 7.8 7.5
7	9.80 912	15	9.92 561	26	0.07 439	9.88 351	11	53	4 10.4 10.0
8	9.80 927	15	9.92 587	25	0.07 413	9.88 340	10	52	5 13.0 12.5
9	9.80 942	15	9.92 612	26	0.07 388	9.88 330	11	51	6 15.6 15.0
10	9.80 957	15	9.92 638	25	0.07 362	9.88 319	11	50	7 18.2 17.5
11	9.80 972	15	9.92 663	26	0.07 337	9.88 308	11	49	8 20.8 20.0
12	9.80 987	15	9.92 689	26	0.07 311	9.88 298	10	48	9 23.4 22.5
13	9.81 002	15	9.92 715	25	0.07 285	9.88 287	11	47	
14	9.81 017	15	9.92 740	26	0.07 260	9.88 276	10	46	
15	9.81 032	15	9.92 766	26	0.07 234	9.88 266	11	45	15 14
16	9.81 047	14	9.92 792	25	0.07 208	9.88 255	11	44	
17	9.81 061	15	9.92 817	26	0.07 183	9.88 244	10	43	1 1.5 1.4
18	9.81 076	15	9.92 843	25	0.07 157	9.88 234	11	42	2 3.0 2.8
19	9.81 091	15	9.92 868	26	0.07 132	9.88 223	11	41	3 4.5 4.2
20	9.81 106	15	9.92 894	26	0.07 106	9.88 212	11	40	4 6.0 5.6
21	9.81 121	15	9.92 920	25	0.07 080	9.88 201	10	39	5 7.5 7.0
22	9.81 136	15	9.92 945	26	0.07 055	9.88 191	11	38	6 9.0 8.4
23	9.81 151	15	9.92 971	25	0.07 029	9.88 180	11	37	7 10.5 9.8
24	9.81 166	14	9.92 996	26	0.07 004	9.88 169	11	36	8 12.0 11.2
25	9.81 180	15	9.93 022	26	0.06 978	9.88 158	10	35	9 13.5 12.6
26	9.81 195	15	9.93 048	25	0.06 952	9.88 148	11	34	
27	9.81 210	15	9.93 073	26	0.06 927	9.88 137	11	33	
28	9.81 225	15	9.93 099	25	0.06 901	9.88 126	11	32	11 10
29	9.81 240	14	9.93 124	26	0.06 876	9.88 115	10	31	
30	9.81 254	15	9.93 150	25	0.06 850	9.88 105	11	30	1 1.1 1.0
31	9.81 269	15	9.93 175	26	0.06 825	9.88 094	11	29	2 2.2 2.0
32	9.81 284	15	9.93 201	26	0.06 799	9.88 083	11	28	3 3.3 3.0
33	9.81 299	15	9.93 227	25	0.06 773	9.88 072	11	27	4 4.4 4.0
34	9.81 314	14	9.93 252	26	0.06 748	9.88 061	11	26	5 5.5 5.0
35	9.81 328	15	9.93 278	25	0.06 722	9.88 051	10	25	6 6.6 6.0
36	9.81 343	15	9.93 303	26	0.06 697	9.88 040	11	24	7 7.7 7.0
37	9.81 358	14	9.93 329	25	0.06 671	9.88 029	11	23	8 8.8 8.0
38	9.81 372	15	9.93 354	26	0.06 646	9.88 018	11	22	9 9.9 9.0
39	9.81 387	15	9.93 380	26	0.06 620	9.88 007	11	21	
40	9.81 402	15	9.93 406	25	0.06 594	9.87 996	11	20	
41	9.81 417	14	9.93 431	26	0.06 569	9.87 985	10	19	
42	9.81 431	15	9.93 457	25	0.06 543	9.87 975	11	18	
43	9.81 446	15	9.93 482	26	0.06 518	9.87 964	11	17	
44	9.81 461	14	9.93 508	25	0.06 492	9.87 953	11	16	11 10 10
45	9.81 475	15	9.93 533	26	0.06 467	9.87 942	11	15	26 26 25
46	9.81 490	15	9.93 559	25	0.06 441	9.87 931	11	14	
47	9.81 505	14	9.93 584	26	0.06 416	9.87 920	11	13	0 1.2 1.3 1.2
48	9.81 519	15	9.93 610	26	0.06 390	9.87 909	11	12	1 3.5 3.9 3.8
49	9.81 534	15	9.93 636	25	0.06 364	9.87 898	11	11	2 5.9 6.5 6.2
50	9.81 549	14	9.93 661	26	0.06 339	9.87 887	10	10	3 8.3 9.1 8.8
51	9.81 563	15	9.93 687	25	0.06 313	9.87 877	11	9	4 10.6 11.7 11.2
52	9.81 578	14	9.93 712	26	0.06 288	9.87 866	11	8	5 13.0 14.3 13.6
53	9.81 592	15	9.93 738	25	0.06 262	9.87 855	11	7	6 15.4 16.9 16.4
54	9.81 607	15	9.93 763	26	0.06 237	9.87 844	11	6	7 17.7 19.5 19.1
55	9.81 622	14	9.93 789	25	0.06 211	9.87 833	11	5	8 20.1 22.1 21.7
56	9.81 636	15	9.93 814	26	0.06 186	9.87 822	11	4	9 22.5 24.8 24.4
57	9.81 651	14	9.93 840	25	0.06 160	9.87 811	11	3	10 24.8
58	9.81 665	15	9.93 865	26	0.06 135	9.87 800	11	2	
59	9.81 680	14	9.93 891	25	0.06 109	9.87 789	11	1	
60	9.81 694		9.93 916		0.06 084	9.87 778		0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

'	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.81 694	15	9.93 916	26	0.06 084	9.87 778	11	60	
1	9.81 709	14	9.93 942	25	0.06 058	9.87 767	11	59	
2	9.81 723	15	9.93 967	26	0.06 033	9.87 756	11	58	
3	9.81 738	14	9.93 993	25	0.06 007	9.87 745	11	57	26 26
4	9.81 752	15	9.94 018	26	0.05 982	9.87 734	11	56	1 2.6 2.5
5	9.81 767	14	9.94 044	25	0.05 956	9.87 723	11	55	2 5.2 5.0
6	9.81 781	15	9.94 069	26	0.05 931	9.87 712	11	54	3 7.8 7.5
7	9.81 796	14	9.94 095	25	0.05 905	9.87 701	11	53	4 10.4 10.0
8	9.81 810	15	9.94 120	26	0.05 880	9.87 690	11	52	5 13.0 12.5
9	9.81 825	14	9.94 146	25	0.05 854	9.87 679	11	51	6 15.6 15.0
10	9.81 839	15	9.94 171	26	0.05 829	9.87 668	11	50	7 18.2 17.5
11	9.81 854	14	9.94 197	25	0.05 803	9.87 657	11	49	8 20.8 20.0
12	9.81 868	15	9.94 222	26	0.05 778	9.87 646	11	48	9 23.4 22.5
13	9.81 882	14	9.94 248	25	0.05 752	9.87 635	11	47	
14	9.81 897	15	9.94 273	26	0.05 727	9.87 624	11	46	
15	9.81 911	14	9.94 299	25	0.05 701	9.87 613	12	45	
16	9.81 926	15	9.94 324	26	0.05 676	9.87 601	11	44	16 14
17	9.81 940	14	9.94 350	25	0.05 650	9.87 590	11	43	1 1.5 1.4
18	9.81 955	15	9.94 375	26	0.05 625	9.87 579	11	42	2 3.0 2.8
19	9.81 969	14	9.94 401	25	0.05 599	9.87 568	11	41	3 4.5 4.2
20	9.81 983	15	9.94 426	26	0.05 574	9.87 557	11	40	4 6.0 5.6
21	9.81 998	14	9.94 452	25	0.05 548	9.87 546	11	39	5 7.5 7.0
22	9.82 012	15	9.94 477	26	0.05 523	9.87 535	11	38	6 9.0 8.4
23	9.82 026	14	9.94 503	25	0.05 497	9.87 524	11	37	7 10.5 9.8
24	9.82 041	15	9.94 528	26	0.05 472	9.87 513	12	36	8 12.0 11.2
25	9.82 055	14	9.94 554	25	0.05 446	9.87 501	11	35	9 13.5 12.6
26	9.82 069	15	9.94 579	26	0.05 421	9.87 490	11	34	
27	9.82 084	14	9.94 604	25	0.05 396	9.87 479	11	33	
28	9.82 098	15	9.94 630	26	0.05 370	9.87 468	11	32	
29	9.82 112	14	9.94 655	25	0.05 345	9.87 457	11	31	12 11
30	9.82 126	15	9.94 681	26	0.05 319	9.87 446	12	30	
31	9.82 141	14	9.94 706	25	0.05 294	9.87 434	11	29	1 1.2 1.1
32	9.82 155	15	9.94 732	26	0.05 268	9.87 423	11	28	2 2.4 2.2
33	9.82 169	14	9.94 757	25	0.05 243	9.87 412	11	27	3 3.6 3.3
34	9.82 184	15	9.94 783	26	0.05 217	9.87 401	11	26	4 4.8 4.4
35	9.82 198	14	9.94 808	25	0.05 192	9.87 390	11	25	5 6.0 5.5
36	9.82 212	15	9.94 834	26	0.05 166	9.87 378	12	24	6 7.2 6.6
37	9.82 226	14	9.94 859	25	0.05 141	9.87 367	11	23	7 8.4 7.7
38	9.82 240	15	9.94 884	26	0.05 116	9.87 356	11	22	8 9.6 8.8
39	9.82 255	14	9.94 910	25	0.05 090	9.87 345	11	21	9 10.8 9.9
40	9.82 269	15	9.94 935	26	0.05 065	9.87 334	12	20	
41	9.82 283	14	9.94 961	25	0.05 039	9.87 322	11	19	
42	9.82 297	15	9.94 986	26	0.05 014	9.87 311	11	18	
43	9.82 311	14	9.95 012	25	0.04 988	9.87 300	12	17	
44	9.82 326	15	9.95 037	26	0.04 963	9.87 288	11	16	12 12 11
45	9.82 340	14	9.95 062	25	0.04 938	9.87 277	11	15	26 25 25
46	9.82 354	15	9.95 088	26	0.04 912	9.87 266	11	14	
47	9.82 368	14	9.95 113	25	0.04 887	9.87 255	12	13	0 1.1 1.0 1.1
48	9.82 382	15	9.95 139	26	0.04 861	9.87 243	12	12	1 3.2 3.1 3.4
49	9.82 396	14	9.95 164	25	0.04 836	9.87 232	11	11	2 5.4 5.2 5.7
50	9.82 410	15	9.95 190	26	0.04 810	9.87 221	12	10	3 7.6 7.3 8.0
..	9.82 424	14	9.95 215	25	0.04 785	9.87 209	11	9	4 9.8 9.4 10.2
	9.82 439	15	9.95 240	26	0.04 760	9.87 198	11	8	5 11.9 11.5 12.5
	9.82 453	14	9.95 266	25	0.04 734	9.87 187	11	7	6 14.1 13.5 14.8
	9.82 467	15	9.95 291	26	0.04 709	9.87 175	12	6	7 16.2 15.6 17.0
	9.82 481	14	9.95 317	25	0.04 683	9.87 164	11	5	8 18.4 17.7 19.3
	9.82 495	15	9.95 342	26	0.04 658	9.87 153	11	4	9 20.6 19.8 21.6
	9.82 509	14	9.95 368	25	0.04 632	9.87 141	12	3	10 22.8 21.9 23.9
	9.82 523	15	9.95 393	26	0.04 607	9.87 130	11	2	11 24.9 24.0 —
	9.82 537	14	9.95 418	25	0.04 582	9.87 119	11	1	
	9.82 551	15	9.95 444	26	0.04 556	9.87 107	12	0	
.. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.	'	P. P.	

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.82 551	14	9.95 444	25	0.04 556	9.87 107	11	60	
1	9.82 565	14	9.95 469	26	0.04 531	9.87 096	11	59	
2	9.82 579	14	9.95 495	26	0.04 505	9.87 085	12	58	
3	9.82 593	14	9.95 520	25	0.04 480	9.87 073	11	57	26 25
4	9.82 607	14	9.95 545	26	0.04 455	9.87 062	12	56	
5	9.82 621	14	9.95 571	25	0.04 429	9.87 050	11	55	1 2.6 2.5
6	9.82 635	14	9.95 596	26	0.04 404	9.87 039	11	54	2 5.2 5.0
7	9.82 649	14	9.95 622	25	0.04 378	9.87 028	12	53	3 7.8 7.5
8	9.82 663	14	9.95 647	25	0.04 353	9.87 016	11	52	4 10.4 10.0
9	9.82 677	14	9.95 672	25	0.04 328	9.87 005	12	51	5 13.0 12.5
10	9.82 691	14	9.95 698	26	0.04 302	9.86 993	11	50	6 15.6 15.0
11	9.82 705	14	9.95 723	25	0.04 277	9.86 982	12	49	7 18.2 17.5
12	9.82 719	14	9.95 748	26	0.04 252	9.86 970	11	48	8 20.8 20.0
13	9.82 733	14	9.95 774	25	0.04 226	9.86 959	12	47	9 23.4 22.5
14	9.82 747	14	9.95 799	26	0.04 201	9.86 947	11	46	
15	9.82 761	14	9.95 825	25	0.04 175	9.86 936	12	45	14 18
16	9.82 775	13	9.95 850	25	0.04 150	9.86 924	11	44	
17	9.82 788	14	9.95 875	26	0.04 125	9.86 913	12	43	1 1.4 1.3
18	9.82 802	14	9.95 901	25	0.04 099	9.86 902	11	42	2 2.8 2.6
19	9.82 816	14	9.95 926	26	0.04 074	9.86 890	12	41	3 4.2 3.9
20	9.82 830	14	9.95 952	25	0.04 048	9.86 879	11	40	4 5.6 5.2
21	9.82 844	14	9.95 977	25	0.04 023	9.86 867	12	39	5 7.0 6.5
22	9.82 858	14	9.96 002	26	0.03 998	9.86 855	11	38	6 8.4 7.8
23	9.82 872	14	9.96 028	26	0.03 972	9.86 844	12	37	7 9.8 9.1
24	9.82 885	13	9.96 053	25	0.03 947	9.86 832	11	36	8 11.2 10.4
25	9.82 899	14	9.96 078	25	0.03 922	9.86 821	12	35	9 12.6 11.7
26	9.82 913	14	9.96 104	26	0.03 896	9.86 809	11	34	
27	9.82 927	14	9.96 129	25	0.03 871	9.86 798	12	33	12 11
28	9.82 941	14	9.96 155	25	0.03 845	9.86 786	11	32	
29	9.82 955	13	9.96 180	25	0.03 820	9.86 775	12	31	1 1.2 1.1
30	9.82 968	14	9.96 205	26	0.03 795	9.86 763	11	30	2 2.4 2.2
31	9.82 982	14	9.96 231	25	0.03 769	9.86 752	12	29	3 3.6 3.3
32	9.82 996	14	9.96 256	25	0.03 744	9.86 740	11	28	4 4.8 4.4
33	9.83 010	14	9.96 281	25	0.03 719	9.86 728	12	27	5 6.0 5.5
34	9.83 023	13	9.96 307	26	0.03 693	9.86 717	11	26	6 7.2 6.6
35	9.83 037	14	9.96 332	25	0.03 668	9.86 705	12	25	7 8.4 7.7
36	9.83 051	14	9.96 357	25	0.03 643	9.86 694	11	24	8 9.6 8.8
37	9.83 065	14	9.96 383	26	0.03 617	9.86 682	12	23	9 10.8 9.9
38	9.83 078	13	9.96 408	25	0.03 592	9.86 670	11	22	
39	9.83 092	14	9.96 433	25	0.03 567	9.86 659	12	21	
40	9.83 106	14	9.96 459	26	0.03 541	9.86 647	11	20	
41	9.83 120	14	9.96 484	25	0.03 516	9.86 635	12	19	
42	9.83 133	13	9.96 510	26	0.03 490	9.86 624	11	18	
43	9.83 147	14	9.96 535	25	0.03 465	9.86 612	12	17	12 11 11
44	9.83 161	14	9.96 560	26	0.03 440	9.86 600	11	16	26 26 25
45	9.83 174	13	9.96 586	26	0.03 414	9.86 589	12	15	
46	9.83 188	14	9.96 611	25	0.03 389	9.86 577	11	14	0 1.1 1.2 1.1
47	9.83 202	14	9.96 636	25	0.03 364	9.86 565	12	13	1 3.2 3.5 3.4
48	9.83 215	13	9.96 662	26	0.03 338	9.86 554	11	12	2 5.4 5.9 5.7
49	9.83 229	14	9.96 687	25	0.03 313	9.86 542	12	11	3 7.6 8.3 8.0
50	9.83 242	13	9.96 712	25	0.03 288	9.86 530	11	10	4 9.8 10.6 10.2
51	9.83 256	14	9.96 738	26	0.03 262	9.86 518	12	9	5 11.9 13.0 12.5
52	9.83 270	14	9.96 763	25	0.03 237	9.86 507	11	8	6 14.1 15.4
53	9.83 283	13	9.96 788	25	0.03 212	9.86 495	12	7	7 16.2 17.7
54	9.83 297	14	9.96 814	26	0.03 186	9.86 483	11	6	8 18.4 20.1
55	9.83 310	13	9.96 839	25	0.03 161	9.86 472	12	5	9 20.6 22.5
56	9.83 324	14	9.96 864	26	0.03 136	9.86 460	11	4	10 22.8
57	9.83 338	14	9.96 890	26	0.03 110	9.86 448	12	3	11 24.9
58	9.83 351	13	9.96 915	25	0.03 085	9.86 436	11	2	
59	9.83 365	14	9.96 940	25	0.03 060	9.86 425	12	1	
60	9.83 378	13	9.96 966	26	0.03 034	9.86 413	11	0	
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. . .

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.83 378	14	9.96 966	25	0.03 034	9.86 413	12	60	
1	9.83 392	14	9.96 991	25	0.03 009	9.86 401	12	59	
2	9.83 405	13	9.97 016	25	0.02 984	9.86 389	12	58	
3	9.83 419	14	9.97 042	25	0.02 958	9.86 377	11	57	28 25
4	9.83 432	13	9.97 067	25	0.02 933	9.86 366	12	56	1 2.6 2.5
5	9.83 446	14	9.97 092	25	0.02 908	9.86 354	12	55	2 5.2 5.0
6	9.83 459	13	9.97 118	26	0.02 882	9.86 342	12	54	3 7.8 7.5
7	9.83 473	14		25	0.02 857	9.86 330	12	53	4 10.4 10.0
8	9.83 486	13	9.97 143	25	0.02 832	9.86 318	12	52	5 13.0 12.5
9	9.83 500	14	9.97 168	25	0.02 807	9.86 306	12	51	6 15.6 15.0
10	9.83 513	13	9.97 193	26	0.02 781	9.86 295	11	50	7 18.2 17.5
11	9.83 527	14	9.97 219	25	0.02 756	9.86 283	12	49	8 20.8 20.0
12	9.83 540	13	9.97 244	25	0.02 731	9.86 271	12	48	9 23.4 22.5
13	9.83 554	14	9.97 269	26	0.02 705	9.86 259	12	47	
14	9.83 567	13	9.97 295	25	0.02 680	9.86 247	12	46	
15	9.83 581	14	9.97 320	25	0.02 655	9.86 235	12	45	14 13
16	9.83 594	13	9.97 345	26	0.02 629	9.86 223	12	44	1 1.4 1.3
17	9.83 608	14	9.97 396	25	0.02 604	9.86 211	11	43	2 2.8 2.6
18	9.83 621	13	9.97 421	25	0.02 579	9.86 200	12	42	3 4.2 3.9
19	9.83 634	14	9.97 447	26	0.02 553	9.86 188	12	41	4 5.6 5.2
20	9.83 648	13	9.97 472	25	0.02 528	9.86 176	12	40	5 7.0 6.5
21	9.83 661	14	9.97 497	25	0.02 503	9.86 164	12	39	6 8.4 7.8
22	9.83 674	13	9.97 523	26	0.02 477	9.86 152	12	38	7 9.8 9.1
23	9.83 688	14	9.97 548	25	0.02 452	9.86 140	12	37	8 11.2 10.4
24	9.83 701	13	9.97 573	25	0.02 427	9.86 128	12	36	9 12.6 11.7
25	9.83 715	14	9.97 598	25	0.02 402	9.86 116	12	35	
26	9.83 728	13	9.97 624	26	0.02 376	9.86 104	12	34	
27	9.83 741	14	9.97 649	25	0.02 351	9.86 092	12	33	13 11
28	9.83 755	13	9.97 674	25	0.02 326	9.86 080	12	32	1 1.2 1.1
29	9.83 768	14	9.97 700	26	0.02 300	9.86 068	12	31	2 2.4 2.2
30	9.83 781	13	9.97 725	25	0.02 275	9.86 056	12	30	3 3.6 3.3
31	9.83 795	14	9.97 750	25	0.02 250	9.86 044	12	29	4 4.8 4.4
32	9.83 808	13	9.97 776	26	0.02 224	9.86 032	12	28	5 6.0 5.5
33	9.83 821	14	9.97 801	25	0.02 199	9.86 020	12	27	6 7.2 6.6
34	9.83 834	13	9.97 826	25	0.02 174	9.86 008	12	26	7 8.4 7.7
35	9.83 848	14	9.97 851	25	0.02 149	9.85 996	12	25	8 9.6 8.8
36	9.83 861	13	9.97 877	26	0.02 123	9.85 984	12	24	9 10.8 9.9
37	9.83 874	14	9.97 902	25	0.02 098	9.85 972	12	23	
38	9.83 887	13	9.97 927	25	0.02 073	9.85 960	12	22	
39	9.83 901	14	9.97 953	26	0.02 047	9.85 948	12	21	
40	9.83 914	13	9.97 978	25	0.02 022	9.85 936	12	20	
41	9.83 927	14	9.98 003	25	0.01 997	9.85 924	12	19	
42	9.83 940	13	9.98 029	26	0.01 971	9.85 912	12	18	
43	9.83 954	14	9.98 054	25	0.01 946	9.85 900	12	17	13 13 12
44	9.83 967	13	9.98 079	25	0.01 921	9.85 888	12	16	28 25 25
45	9.83 980	14	9.98 104	25	0.01 896	9.85 876	12	15	0 1.0 1.0
46	9.83 993	13	9.98 130	26	0.01 870	9.85 864	12	14	1 3.0 2.9
47	9.84 006	14	9.98 155	25	0.01 845	9.85 851	13	13	2 5.0 4.8
48	9.84 020	13	9.98 180	25	0.01 820	9.85 839	12	12	3 7.0 6.7
49	9.84 033	14	9.98 206	26	0.01 794	9.85 827	12	11	4 9.0 8.7
50	9.84 046	13	9.98 231	25	0.01 769	9.85 815	12	10	5 11.0 10.6
51	9.84 059	14	9.98 256	25	0.01 744	9.85 803	12	9	6 13.0 12.5
	9.84 072	13	9.98 281	25	0.01 719	9.85 791	12	8	7 15.0 14.4
	9.84 085	14	9.98 307	26	0.01 693	9.85 779	12	7	8 17.0 16.3
	9.84 098	13	9.98 332	25	0.01 668	9.85 766	13	6	9 19.0 18.3
	9.84 112	14	9.98 357	25	0.01 643	9.85 754	12	5	10 21.0 20.2
	9.84 125	13	9.98 383	26	0.01 617	9.85 742	12	4	11 23.0 22.1
	9.84 138	14	9.98 408	25	0.01 592	9.85 730	12	3	12 25.0 24.0
	9.84 151	13	9.98 433	25	0.01 567	9.85 718	12	2	13 —
	9.84 164	14	9.98 458	25	0.01 542	9.85 706	12	1	
	9.84 177	13	9.98 484	26	0.01 516	9.85 693	13	0	
	Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

	L. Sin.	d.	L. Tan.	c. d.	L. Cot.	L. Cos.	d.		P. P.
0	9.84 177	13	9.98 484	25	0.01 516	9.85 693	12	60	
1	9.84 190	13	9.98 509	25	0.01 491	9.85 681	12	59	
2	9.84 203	13	9.98 534	26	0.01 466	9.85 669	12	58	
3	9.84 216	13	9.98 560	25	0.01 440	9.85 657	12	57	26 25 14
4	9.84 229	13	9.98 585	25	0.01 415	9.85 645	12	56	2 2.6 2.5 1.4
5	9.84 242	13	9.98 610	25	0.01 390	9.85 632	12	55	3 5.2 5.0 2.8
6	9.84 255	14	9.98 635	26	0.01 365	9.85 620	12	54	4 7.8 7.5 4.2
7	9.84 269	13	9.98 661	25	0.01 339	9.85 608	12	53	4 10.4 10.0 5.6
8	9.84 282	13	9.98 686	25	0.01 314	9.85 596	12	52	5 13.0 12.5 7.0
9	9.84 295	13	9.98 711	26	0.01 289	9.85 583	12	51	6 15.6 15.0 8.4
10	9.84 308	13	9.98 737	25	0.01 263	9.85 571	12	50	7 18.2 17.5 9.8
11	9.84 321	13	9.98 762	25	0.01 238	9.85 559	12	49	8 20.8 20.0 11.2
12	9.84 334	13	9.98 787	25	0.01 213	9.85 547	12	48	9 23.4 22.5 12.6
13	9.84 347	13	9.98 812	26	0.01 188	9.85 534	12	47	
14	9.84 360	13	9.98 838	25	0.01 162	9.85 522	12	46	13 12
15	9.84 373	12	9.98 863	25	0.01 137	9.85 510	12	45	1 1.3 1.2
16	9.84 385	13	9.98 888	25	0.01 112	9.85 497	12	44	2 2.6 2.4
17	9.84 398	13	9.98 913	26	0.01 087	9.85 485	12	43	3 3.9 3.6
18	9.84 411	13	9.98 939	25	0.01 061	9.85 473	12	42	4 5.2 4.8
19	9.84 424	13	9.98 964	25	0.01 036	9.85 460	12	41	5 6.5 6.0
20	9.84 437	13	9.98 989	26	0.01 011	9.85 448	12	40	6 7.8 7.2
21	9.84 450	13	9.99 015	25	0.00 985	9.85 436	12	39	7 9.1 8.4
22	9.84 463	13	9.99 040	25	0.00 960	9.85 423	12	38	8 10.4 9.6
23	9.84 476	13	9.99 065	25	0.00 935	9.85 411	12	37	9 11.7 10.8
24	9.84 489	13	9.99 090	26	0.00 910	9.85 399	12	36	
25	9.84 502	13	9.99 116	25	0.00 884	9.85 386	12	35	
26	9.84 515	13	9.99 141	25	0.00 859	9.85 374	12	34	
27	9.84 528	12	9.99 166	25	0.00 834	9.85 361	12	33	18 18
28	9.84 540	13	9.99 191	26	0.00 809	9.85 349	12	32	26 25
29	9.84 553	13	9.99 217	25	0.00 783	9.85 337	12	31	
30	9.84 566	13	9.99 242	25	0.00 758	9.85 324	12	30	0 1.0 1.0
31	9.84 579	13	9.99 267	26	0.00 733	9.85 312	12	29	1 3.0 2.9
32	9.84 592	13	9.99 293	25	0.00 707	9.85 299	12	28	2 5.0 4.8
33	9.84 605	13	9.99 318	25	0.00 682	9.85 287	12	27	3 7.0 6.7
34	9.84 618	12	9.99 343	25	0.00 657	9.85 274	12	26	4 9.0 8.7
35	9.84 630	13	9.99 368	26	0.00 632	9.85 262	12	25	5 11.0 10.6
36	9.84 643	13	9.99 394	25	0.00 606	9.85 250	12	24	6 13.0 12.5
37	9.84 656	13	9.99 419	25	0.00 581	9.85 237	12	23	7 15.0 14.4
38	9.84 669	13	9.99 444	25	0.00 556	9.85 225	12	22	8 17.0 16.3
39	9.84 682	12	9.99 469	26	0.00 531	9.85 212	12	21	9 19.0 18.3
40	9.84 694	13	9.99 495	25	0.00 505	9.85 200	12	20	10 21.0 20.2
41	9.84 707	13	9.99 520	25	0.00 480	9.85 187	12	19	11 23.0 22.1
42	9.84 720	13	9.99 545	25	0.00 455	9.85 175	12	18	12 25.0 24.0
43	9.84 733	12	9.99 570	26	0.00 430	9.85 162	12	17	
44	9.84 745	13	9.99 596	25	0.00 404	9.85 150	12	16	
45	9.84 758	13	9.99 621	25	0.00 379	9.85 137	12	15	12 12
46	9.84 771	13	9.99 646	26	0.00 354	9.85 125	12	14	26 25
47	9.84 784	12	9.99 672	25	0.00 328	9.85 112	12	13	
48	9.84 796	13	9.99 697	25	0.00 303	9.85 100	12	12	0 1.1 1.0
49	9.84 809	13	9.99 722	25	0.00 278	9.85 087	12	11	1 3.2 3.1
50	9.84 822	13	9.99 747	26	0.00 253	9.85 074	12	10	2 5.4 5.2
51	9.84 835	12	9.99 773	25	0.00 227	9.85 062	12	9	3 7.6 7.3
52	9.84 847	13	9.99 798	25	0.00 202	9.85 049	12	8	4 9.8 9.4
53	9.84 860	13	9.99 823	25	0.00 177	9.85 037	12	7	5 14.1
54	9.84 873	12	9.99 848	26	0.00 152	9.85 024	12	6	6 16.2
55	9.84 885	13	9.99 874	25	0.00 126	9.85 012	12	5	7 18.4
56	9.84 898	13	9.99 899	25	0.00 101	9.84 999	12	4	8 1
57	9.84 911	12	9.99 924	25	0.00 076	9.84 986	12	3	9 14.1
58	9.84 923	13	9.99 949	26	0.00 051	9.84 974	12	2	10 16.2
59	9.84 936	13	9.99 975	25	0.00 025	9.84 961	12	1	11 18.4
60	9.84 949	13	0.00 000	25	0.00 000	9.84 949	12	0	12 1
	L. Cos.	d.	L. Cot.	c. d.	L. Tan.	L. Sin.	d.		P. P.

III.
NATURAL
TRIGONOMETRIC FUNCTIONS
FOR EACH MINUTE.

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.00000	.00000	∞	1.0000	60
1	.029	.029	3437.7	.000	59
2	.058	.058	1718.9	.000	58
3	.087	.087	1145.9	.000	57
4	.116	.116	859.44	.000	56
5	.00145	.00145	687.55	1.0000	55
6	.175	.175	572.96	.000	54
7	.204	.204	491.11	.000	53
8	.233	.233	429.72	.000	52
9	.262	.262	381.97	.000	51
10	.00291	.00291	343.77	1.0000	50
11	.320	.320	312.52	.99999	49
12	.349	.349	286.48	.999	48
13	.378	.378	264.44	.999	47
14	.407	.407	245.55	.999	46
15	.00436	.00436	229.18	.99999	45
16	.465	.465	214.86	.999	44
17	.495	.495	202.22	.999	43
18	.524	.524	190.98	.999	42
19	.553	.553	180.93	.998	41
20	.00582	.00582	171.89	.99998	40
21	.611	.611	163.70	.998	39
22	.640	.640	156.26	.998	38
23	.669	.669	149.47	.998	37
24	.698	.698	143.24	.998	36
25	.00727	.00727	137.51	.99997	35
26	.756	.756	132.22	.997	34
27	.785	.785	127.32	.997	33
28	.814	.815	122.77	.997	32
29	.844	.844	118.54	.996	31
30	.00873	.00873	114.59	.99996	30
31	.902	.902	110.89	.996	29
32	.931	.931	107.43	.996	28
33	.960	.960	104.17	.995	27
34	.00989	.00989	101.11	.995	26
35	.01018	.01018	98.218	.99995	25
36	.047	.047	95.489	.995	24
37	.076	.076	92.908	.994	23
38	.105	.105	90.463	.994	22
39	.134	.135	88.144	.994	21
40	.01164	.01164	85.940	.99993	20
41	.193	.193	83.844	.993	19
42	.222	.222	81.847	.993	18
43	.251	.251	79.943	.992	17
44	.280	.280	78.126	.992	16
45	.01309	.01309	76.390	.99991	15
46	.338	.338	74.729	.991	14
47	.367	.367	73.139	.991	13
48	.396	.396	71.615	.990	12
49	.425	.425	70.153	.990	11
50	.01454	.01455	68.750	.99989	10
51	.483	.484	67.402	.989	9
52	.513	.513	66.105	.989	8
53	.542	.542	64.858	.988	7
54	.571	.571	63.657	.988	6
55	.01600	.01600	62.499	.99987	5
56	.629	.629	61.383	.987	4
57	.658	.658	60.306	.986	3
58	.687	.687	59.266	.986	2
59	.716	.716	58.261	.985	1
60	.01745	.01746	57.290	.99985	0
N. Cos.		N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.01745	.01746	57.290	.99985	60
1	.774	.775	56.351	.984	59
2	.803	.804	55.442	.984	58
3	.832	.833	54.561	.983	57
4	.862	.862	53.709	.983	56
5	.01891	.01891	52.882	.99982	55
6	.920	.920	52.081	.982	54
7	.949	.949	51.303	.981	53
8	.01978	.01978	50.549	.980	52
9	.02007	.02007	49.816	.980	51
10	.02036	.02036	49.104	.99979	50
11	.065	.066	48.412	.979	49
12	.094	.095	47.740	.978	48
13	.123	.124	47.085	.977	47
14	.152	.153	46.449	.977	46
15	.02181	.02182	45.829	.99976	45
16	.211	.211	45.226	.976	44
17	.240	.240	44.639	.975	43
18	.269	.269	44.066	.974	42
19	.298	.298	43.508	.974	41
20	.02327	.02328	42.964	.99973	40
21	.356	.357	42.433	.972	39
22	.385	.386	41.916	.972	38
23	.414	.415	41.411	.971	37
24	.443	.444	40.917	.970	36
25	.02472	.02473	40.436	.99969	35
26	.501	.502	39.965	.969	34
27	.530	.531	39.506	.968	33
28	.560	.560	39.057	.967	32
29	.589	.589	38.618	.966	31
30	.02618	.02619	38.188	.99966	30
31	.647	.648	37.769	.965	29
32	.676	.677	37.358	.964	28
33	.705	.706	36.956	.963	27
34	.734	.735	36.563	.963	26
35	.02763	.02764	36.178	.99962	25
36	.792	.793	35.801	.961	24
37	.821	.822	35.431	.960	23
38	.850	.851	35.070	.959	22
39	.879	.881	34.715	.959	21
40	.02908	.02910	34.368	.99958	20
41	.938	.939	34.027	.957	19
42	.967	.968	33.694	.956	18
43	.02996	.02997	33.366	.955	17
44	.03025	.03026	33.045	.954	16
45	.03054	.03055	32.730	.99953	15
46	.083	.084	32.421	.952	14
47	.112	.114	32.118	.952	13
48	.141	.143	31.821	.951	12
49	.170	.172	31.528	.950	11
50	.03199	.03201	31.242	.99949	10
51	.228	.230	30.960	.948	9
52	.257	.259	30.6		
53	.286	.288	30.4		
54	.316	.317	30.145		
55	.03345	.03346	29.8		
56	.374	.376	29.6		
57	.403	.405	29.3		
58	.432	.434	29.1		
59	.461	.463	28.877	.940	
60	.03490	.03492	28.636	.99939	
N. Cos.		N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.03490	.03492	28.636	.99939	60
1	519	521	.399	938	59
2	548	550	28.166	937	58
3	577	579	27.937	936	57
4	606	609	.712	935	56
5	.03635	.03638	27.490	.99934	55
6	664	667	.271	933	54
7	693	696	27.057	932	53
8	723	725	26.845	931	52
9	752	754	.637	930	51
10	.03781	.03783	26.432	.99929	50
11	810	812	.230	927	49
12	839	842	26.031	926	48
13	868	871	25.835	925	47
14	897	900	.642	924	46
15	.03926	.03929	25.452	.99923	45
16	955	958	.264	922	44
17	.03984	.03987	25.080	921	43
18	.04013	.04016	24.898	919	42
19	042	046	.719	918	41
20	.04071	.04075	24.542	.99917	40
21	100	104	.368	916	39
22	129	133	.196	915	38
23	159	162	24.026	913	37
24	188	191	23.859	912	36
25	.04217	.04220	23.695	.99911	35
26	246	250	.532	910	34
27	275	279	.372	909	33
28	304	308	.214	907	32
29	333	337	23.058	906	31
30	.04362	.04366	22.904	.99905	30
31	391	395	.752	904	29
32	420	424	.602	902	28
33	449	454	.454	901	27
34	478	483	.308	900	26
35	.04507	.04512	22.104	.99898	25
36	536	541	22.022	897	24
37	565	570	21.881	896	23
38	594	599	.743	894	22
39	623	628	.606	893	21
40	.04653	.04658	21.470	.99892	20
41	682	687	.337	890	19
42	711	716	.205	889	18
43	740	745	21.075	888	17
44	769	774	20.946	886	16
45	.04798	.04803	20.819	.99885	15
46	827	833	.693	883	14
47	856	862	.569	882	13
48	885	891	.446	881	12
49	914	920	.325	879	11
50	.04943	.04949	20.206	.99878	10
51	.04972	.04978	20.087	876	9
		7	19.970	875	8
		7	.855	873	7
		.5	.740	872	6
		12	19.627	.99870	5
			.516	869	4
			.405	867	3
			.296	866	2
			.188	864	1
234	.05241	19.081	.99863		0
Cos. N. Cot. N. Tan. N. Sin.					

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.05234	.05241	19.081	.99863	60
1	263	270	18.976	861	59
2	292	299	.871	860	58
3	321	328	.768	858	57
4	350	357	.666	857	56
5	.05379	.05387	18.564	.99855	55
6	408	416	.464	854	54
7	437	445	.366	852	53
8	466	474	.268	851	52
9	495	503	.171	849	51
10	.05524	.05533	18.075	.99847	50
11	553	562	17.980	846	49
12	582	591	.886	844	48
13	611	620	.793	842	47
14	640	649	.702	841	46
15	.05669	.05678	17.611	.99839	45
16	698	708	.521	838	44
17	727	737	.431	836	43
18	756	766	.343	834	42
19	785	795	.256	833	41
20	.05814	.05824	17.169	.99831	40
21	844	854	17.084	829	39
22	873	883	16.999	827	38
23	902	912	.915	826	37
24	931	941	.832	824	36
25	.05960	.05970	16.750	.99822	35
26	.05989	.05999	.608	821	34
27	.06018	.06029	.587	819	33
28	047	058	.507	817	32
29	076	087	.428	815	31
30	.06105	.06116	16.350	.99813	30
31	134	145	.272	812	29
32	163	175	.195	810	28
33	192	204	.119	808	27
34	221	233	16.043	806	26
35	.06250	.06262	15.969	.99804	25
36	279	291	.895	803	24
37	308	321	.821	801	23
38	337	350	.748	799	22
39	366	379	.676	797	21
40	.06395	.06408	15.605	.99795	20
41	424	438	.534	793	19
42	453	467	.464	792	18
43	482	496	.394	790	17
44	511	525	.325	788	16
45	.06540	.06554	15.257	.99786	15
46	569	584	.189	784	14
47	598	613	.122	782	13
48	627	642	15.056	780	12
49	656	671	14.990	778	11
50	.06685	.06700	14.924	.99776	10
51	714	730	.860	774	9
52	743	759	.795	772	8
53	773	788	.732	770	7
54	802	817	.669	768	6
55	.06831	.06847	14.606	.99766	5
56	860	876	.544	764	4
57	889	905	.482	762	3
58	918	934	.421	760	2
59	947	963	.361	758	1
60	.06976	.06993	14.301	.99756	0
N. Cos. N. Cot. N. Tan. N. Sin.					

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.06976	.06993	14.301	.99756	60
1	.07005	.07022	.241	754	59
2	034	051	.182	752	58
3	063	080	.124	750	57
4	092	110	.065	748	56
5	.07121	.07139	14.008	.99746	55
6	150	168	13.951	744	54
7	179	197	.894	742	53
8	208	227	.838	740	52
9	237	256	.782	738	51
10	.07266	.07285	13.727	.99736	50
11	295	314	.672	734	49
12	324	344	.617	731	48
13	353	373	.563	729	47
14	382	402	.510	727	46
15	.07411	.07431	13.457	.99725	45
16	440	461	.404	723	44
17	469	490	.352	721	43
18	498	519	.300	719	42
19	527	548	.248	716	41
20	.07556	.07578	13.197	.99714	40
21	585	607	.146	712	39
22	614	636	.096	710	38
23	643	665	13.046	708	37
24	672	695	12.996	705	36
25	.07701	.07724	12.947	.99703	35
26	730	753	.898	701	34
27	759	782	.850	699	33
28	788	812	.801	696	32
29	817	841	.754	694	31
30	.07846	.07870	12.706	.99692	30
31	875	899	.659	689	29
32	904	929	.612	687	28
33	933	958	.566	685	27
34	962	.07987	.520	683	26
35	.07991	.08017	12.474	.99680	25
36	.08020	046	.429	678	24
37	049	075	.384	676	23
38	078	104	.339	673	22
39	107	134	.295	671	21
40	.08136	.08163	12.251	.99668	20
41	165	192	.207	666	19
42	194	221	.163	664	18
43	223	251	.120	661	17
44	252	280	.077	659	16
45	.08281	.08309	12.035	.99657	15
46	310	339	11.992	654	14
47	339	368	.950	652	13
48	368	397	.909	649	12
49	397	427	.867	647	11
50	.08426	.08456	11.826	.99644	10
51	455	485	.785	642	9
52	484	514	.745	639	8
53	513	544	.705	637	7
54	542	573	.664	635	6
55	.08571	.08602	11.625	.99632	5
56	600	632	.585	630	4
57	629	661	.546	627	3
58	658	690	.507	625	2
59	687	720	.468	622	1
60	.08716	.08749	11.430	.99619	0
N. Cos. N. Cot. N. Tan. N. Sin.					

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.08716	.08749	11.430	.99619	60
1	745	778	.392	617	59
2	774	807	.354	614	58
3	803	837	.316	612	57
4	831	866	.279	609	56
5	.08860	.08895	11.242	.99607	55
6	889	925	.205	604	54
7	918	954	.168	602	53
8	947	.08983	.132	599	52
9	.08976	.09013	.095	596	51
10	.09005	.09042	11.059	.99594	50
11	034	071	11.024	591	49
12	063	101	10.988	588	48
13	092	130	.953	586	47
14	121	159	.918	583	46
15	.09150	.09189	10.883	.99580	45
16	179	218	.848	578	44
17	208	247	.814	575	43
18	237	277	.780	572	42
19	266	306	.746	570	41
20	.09295	.09335	10.712	.99567	40
21	324	365	.678	564	39
22	353	394	.645	562	38
23	382	423	.612	559	37
24	411	453	.579	556	36
25	.09440	.09482	10.546	.99553	35
26	469	511	.514	551	34
27	498	541	.481	548	33
28	527	570	.449	545	32
29	556	600	.417	542	31
30	.09585	.09629	10.385	.99540	30
31	614	658	.354	537	29
32	642	688	.322	534	28
33	671	717	.291	531	27
34	700	746	.260	528	26
35	.09729	.09776	10.229	.99526	25
36	758	805	.199	523	24
37	787	834	.168	520	23
38	816	864	.138	517	22
39	845	893	.108	514	21
40	.09874	.09923	10.078	.99511	20
41	903	952	.048	508	19
42	932	.09981	10.019	506	18
43	961	1.0011	9.9893	503	17
44	.09990	040	.9601	500	16
45	.10019	.10069	9.9310	.99497	15
46	048	099	.9021	494	14
47	077	128	.8734	491	13
48	106	158	.8448	488	12
49	135	187	.8164	485	11
50	.10164	.10216	9.7882	.99482	10
51	192	246	.7601	479	9
52	221	275	.7322	476	8
53	250	305	.7044	473	7
54	279	334	.6768	470	6
55	.10308	.10363	9.6493	.99467	5
56	337	393	.6220	464	4
57	366	422	.5949	461	3
58	395	452	.5679	458	2
59	424	481	.5411	455	1
60	.10453	.10510	9.5144	.99452	0
N. Cos. N. Cot. N. Tan. N. Sin.					

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.10453	.10510	9.5144	.99452	60
1	482	540	.4878	449	59
2	511	569	.4614	446	58
3	540	599	.4352	443	57
4	569	628	.4090	440	56
5	.10597	.10657	9.3831	.99437	55
6	626	687	.3572	434	54
7	655	716	.3315	431	53
8	684	746	.3060	428	52
9	713	775	.2806	424	51
10	.10742	.10805	9.2553	.99421	50
11	771	834	.2302	418	49
12	800	863	.2052	415	48
13	829	893	.1803	412	47
14	858	922	.1555	409	46
15	.10887	.10952	9.1309	.99406	45
16	916	.10981	.1065	402	44
17	945	.11011	.0821	399	43
18	.10973	.040	.0579	396	42
19	.11002	.070	.0338	393	41
20	.11031	.11099	9.0098	.99390	40
21	060	128	8.9860	386	39
22	089	158	.9623	383	38
23	118	187	.9387	380	37
24	147	217	.9152	377	36
25	.11176	.11246	8.8919	.99374	35
26	205	276	.8686	370	34
27	234	305	.8455	367	33
28	263	335	.8225	364	32
29	291	364	.7996	360	31
30	.11320	.11394	8.7769	.99357	30
31	349	423	.7542	354	29
32	378	452	.7317	351	28
33	407	482	.7093	347	27
34	436	511	.6870	344	26
35	.11465	.11541	8.6648	.99341	25
36	494	570	.6427	337	24
37	523	600	.6208	334	23
38	552	629	.5989	331	22
39	580	659	.5772	327	21
40	.11609	.11688	8.5555	.99324	20
41	638	718	.5340	320	19
42	667	747	.5126	317	18
43	696	777	.4913	314	17
44	725	806	.4701	310	16
45	.11754	.11836	8.4490	.99307	15
46	783	865	.4280	303	14
47	812	895	.4071	300	13
48	840	924	.3863	297	12
49	869	954	.3656	293	11
50	.11898	.11983	8.3450	.99290	10
51	927	.12013	.3245	286	9
	956	.042	.3041	283	8
	.11985	.072	.2838	279	7
	.12014	.101	.2636	276	6
	.12043	.12131	8.2434	.99272	5
	071	.160	.2234	269	4
	100	.190	.2035	265	3
	129	.219	.1837	262	2
	158	.249	.1640	258	1
	.12187	.12278	8.1443	.99255	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.12187	.12278	8.1443	.99255	60
1	216	308	.1248	251	59
2	245	338	.1054	248	58
3	274	367	.0860	244	57
4	302	397	.0667	240	56
5	.12331	.12426	8.0476	.99237	55
6	360	456	.0285	233	54
7	389	485	8.0095	230	53
8	418	515	7.9906	226	52
9	447	544	.9718	222	51
10	.12474	.12574	7.9530	.99219	50
11	504	603	.9344	215	49
12	533	633	.9158	211	48
13	562	662	.8973	208	47
14	591	692	.8789	204	46
15	.12620	.12722	7.8606	.99200	45
16	649	751	.8424	197	44
17	678	781	.8243	193	43
18	706	810	.8062	189	42
19	735	840	.7882	186	41
20	.12764	.12869	7.7704	.99182	40
21	793	899	.7525	178	39
22	822	929	.7348	175	38
23	851	958	.7171	171	37
24	880	.12988	.6996	167	36
25	.12908	.13017	7.6821	.99163	35
26	937	047	.6647	160	34
27	966	076	.6473	156	33
28	.12995	106	.6301	152	32
29	.13024	136	.6129	148	31
30	.13053	.13165	7.5958	.99144	30
31	081	195	.5787	141	29
32	110	224	.5618	137	28
33	139	254	.5449	133	27
34	168	284	.5281	129	26
35	.13197	.13313	7.5113	.99125	25
36	226	343	.4947	122	24
37	254	372	.4781	118	23
38	283	402	.4615	114	22
39	312	432	.4451	110	21
40	.13341	.13461	7.4287	.99106	20
41	370	491	.4124	102	19
42	399	521	.3962	098	18
43	427	550	.3800	094	17
44	456	580	.3639	091	16
45	.13485	.13609	7.3479	.99087	15
46	514	639	.3319	083	14
47	543	669	.3160	079	13
48	572	698	.3002	075	12
49	600	728	.2844	071	11
50	.13629	.13758	7.2687	.99067	10
51	658	787	.2531	063	9
52	687	817	.2375	059	8
53	716	846	.2220	055	7
54	744	876	.2066	051	6
55	.13773	.13906	7.1912	.99047	5
56	802	935	.1759	043	4
57	831	965	.1607	039	3
58	860	.13995	.1455	035	2
59	889	.14024	.1304	031	1
60	.13917	.14054	7.1154	.99027	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.13917	.14054	7.1154	.99027	60
1	946	084	.1004	023	59
2	.13975	113	.0855	019	58
3	.14004	143	.0706	015	57
4	033	173	.0558	011	56
5	.14061	.14202	7.0410	.99006	55
6	090	232	.0264	.99002	54
7	119	262	7.0117	.98998	53
8	148	291	6.9972	994	52
9	177	321	.9827	990	51
10	.14205	.14351	6.9682	.98986	50
11	234	381	.9538	982	49
12	263	410	.9395	978	48
13	292	440	.9252	973	47
14	320	470	.9110	969	46
15	.14349	.14499	6.8969	.98965	45
16	378	529	.8828	961	44
17	407	559	.8687	957	43
18	436	588	.8548	953	42
19	464	618	.8408	948	41
20	.14493	.14648	6.8269	.98944	40
21	522	678	.8131	940	39
22	551	707	.7994	936	38
23	580	737	.7856	931	37
24	608	767	.7720	927	36
25	.14637	.14796	6.7584	.98923	35
26	666	826	.7448	919	34
27	695	856	.7313	914	33
28	723	886	.7179	910	32
29	752	915	.7045	906	31
30	.14781	.14945	6.6912	.98902	30
31	810	.14975	.6779	897	29
32	838	.15005	.6646	893	28
33	867	034	.6514	889	27
34	896	064	.6383	884	26
35	.14925	.15094	6.6252	.98880	25
36	954	124	.6122	876	24
37	.14982	153	.5992	871	23
38	.15011	183	.5863	867	22
39	040	213	.5734	863	21
40	.15069	.15243	6.5606	.98858	20
41	097	272	.5478	854	19
42	126	302	.5350	849	18
43	155	332	.5223	845	17
44	184	362	.5097	841	16
45	.15212	.15391	6.4971	.98836	15
46	241	421	.4846	832	14
47	270	451	.4721	827	13
48	299	481	.4596	823	12
49	327	511	.4472	818	11
50	.15356	.15540	6.4348	.98814	10
51	385	570	.4225	809	9
52	414	600	.4103	805	8
53	442	630	.3980	800	7
54	471	660	.3859	796	6
55	.15500	.15689	6.3737	.98791	5
56	529	719	.3617	787	4
57	557	749	.3496	782	3
58	586	779	.3376	778	2
59	615	809	.3257	773	1
60	.15643	.15838	6.3138	.98769	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.15643	.15838	6.3138	.98769	60
1	672	868	.3019	764	59
2	701	898	.2901	760	58
3	730	928	.2783	755	57
4	758	958	.2666	751	56
5	.15787	.15988	6.2549	.98746	55
6	816	.16017	.2432	741	54
7	845	047	.2316	737	53
8	873	077	.2200	732	52
9	902	107	.2085	728	51
10	.15931	.16137	6.1970	.98723	50
11	959	167	.1856	718	49
12	.15988	196	.1742	714	48
13	.16017	226	.1628	709	47
14	046	256	.1515	704	46
15	.16074	.16286	6.1402	.98700	45
16	103	316	.1290	695	44
17	132	346	.1178	690	43
18	160	376	.1066	686	42
19	189	405	.0955	681	41
20	.16218	.16435	6.0844	.98676	40
21	246	465	.0734	671	39
22	275	495	.0624	667	38
23	304	525	.0514	662	37
24	333	555	.0405	657	36
25	.16361	.16585	6.0296	.98652	35
26	390	615	.0188	648	34
27	419	645	6.0080	643	33
28	447	674	5.9972	638	32
29	476	704	.9865	633	31
30	.16505	.16734	5.9758	.98629	30
31	533	764	.9651	624	29
32	562	794	.9545	619	28
33	591	824	.9439	614	27
34	620	854	.9333	609	26
35	.16648	.16884	5.9228	.98604	25
36	677	914	.9124	600	24
37	706	944	.9019	595	23
38	734	.16974	.8915	590	22
39	763	.17004	.8811	585	21
40	.16792	.17033	5.8708	.98580	20
41	820	063	.8605	575	19
42	849	093	.8502	570	18
43	878	123	.8400	565	17
44	906	153	.8298	561	16
45	.16935	.17183	5.8197	.98556	15
46	964	213	.8095	551	14
47	.16992	243	.7994	546	13
48	.17021	273	.7894	541	12
49	050	303	.7794	536	11
50	.17078	.17333	5.7694	.98531	10
51	107	363	.7594	526	9
52	136	393	.7495	521	
53	164	423	.7396	516	
54	193	453	.7297		
55	.17222	.17483	5.7199	.98	
56	250	513	.7101		
57	279	543	.7004		
58	308	573	.6906		
59	336	603	.6809		
60	.17365	.17633	5.6713	.98481	
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.17363	.17633	.56713	.98481	60
1	393	663	.6617	476	59
2	422	693	.6521	471	58
3	451	723	.6425	466	57
4	479	753	.6329	461	56
5	.17508	.17783	.56234	.98455	55
6	537	813	.6140	450	54
7	565	843	.6045	445	53
8	594	873	.5951	440	52
9	623	903	.5857	435	51
10	.17651	.17933	.55764	.98430	50
11	680	963	.5671	425	49
12	708	.17993	.5578	420	48
13	737	.18023	.5485	414	47
14	766	053	.5393	409	46
15	.17794	.18083	.55301	.98404	45
16	823	113	.5209	399	44
17	852	143	.5118	394	43
18	880	173	.5026	389	42
19	909	203	.4936	383	41
20	.17937	.18233	.54845	.98378	40
21	966	263	.4755	373	39
22	.17995	293	.4665	368	38
23	.18023	323	.4575	362	37
24	052	353	.4486	357	36
25	.18081	.18384	.54397	.98352	35
26	109	414	.4308	347	34
27	138	444	.4219	341	33
28	166	474	.4131	336	32
29	195	504	.4043	331	31
30	.18224	.18534	.53955	.98325	30
31	252	564	.3868	320	29
32	281	594	.3781	315	28
33	309	624	.3694	310	27
34	338	654	.3607	304	26
35	.18367	.18684	.53521	.98299	25
36	395	714	.3435	294	24
37	424	745	.3349	288	23
38	452	775	.3263	283	22
39	481	805	.3178	277	21
40	.18509	.18835	.53993	.98272	20
41	538	865	.3008	267	19
42	567	895	.2924	261	18
43	595	925	.2839	256	17
44	624	955	.2755	250	16
45	.18652	.18986	.52672	.98245	15
46	681	.19016	.2588	240	14
47	710	046	.2505	234	13
48	738	076	.2422	229	12
49	767	106	.2339	223	11
50	.18795	.19136	.52257	.98218	10
51	795	136	.22174	212	9
52	823	166	.2092	207	8
53	852	195	.2011	201	7
54	880	224	.1929	196	6
55	.19024	.19287	.51848	.98190	5
56	909	253	.1767	185	4
57	937	282	.1686	179	3
58	966	311	.1606	174	2
59	995	340	.1526	168	1
60	.19081	.19438	.51446	.98163	0
Cos. N. Cot. N. Tan. N. Sin.					

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.19081	.19438	.51446	.98163	60
1	109	468	.1366	157	59
2	138	498	.1286	152	58
3	167	529	.1207	146	57
4	195	559	.1128	140	56
5	.19224	.19589	.51049	.98135	55
6	252	619	.0970	129	54
7	281	649	.0892	124	53
8	309	680	.0814	118	52
9	338	710	.0736	112	51
10	.19366	.19740	.50658	.98107	50
11	395	770	.0581	101	49
12	423	801	.0504	096	48
13	452	831	.0427	090	47
14	481	861	.0350	084	46
15	.19509	.19891	.50273	.98079	45
16	538	921	.0197	073	44
17	566	952	.0121	067	43
18	595	.19982	.50045	061	42
19	623	.20012	.49969	056	41
20	.19652	.20042	.49894	.98050	40
21	680	073	.9819	044	39
22	709	103	.9744	039	38
23	737	133	.9669	033	37
24	766	164	.9594	027	36
25	.19794	.20194	.49520	.98021	35
26	823	224	.9446	016	34
27	851	254	.9372	010	33
28	880	285	.9298	.98004	32
29	908	315	.9225	.97998	31
30	.19937	.20345	.49152	.97992	30
31	965	376	.9078	987	29
32	.19994	406	.9006	981	28
33	.20022	436	.8933	975	27
34	051	466	.8860	969	26
35	.20079	.20497	.48788	.97963	25
36	108	527	.8716	958	24
37	136	557	.8644	952	23
38	165	588	.8573	946	22
39	193	618	.8501	940	21
40	.20222	.20648	.48430	.97934	20
41	250	679	.8359	928	19
42	279	709	.8288	922	18
43	307	739	.8218	916	17
44	336	770	.8147	910	16
45	.20364	.20800	.48077	.97905	15
46	393	830	.8007	899	14
47	421	861	.7937	893	13
48	450	891	.7867	887	12
49	478	921	.7798	881	11
50	.20507	.20952	.47729	.97875	10
51	535	.20982	.7659	869	9
52	563	.21013	.7591	863	8
53	592	043	.7522	857	7
54	620	073	.7453	851	6
55	.20649	.21104	.47385	.97845	5
56	677	134	.7317	839	4
57	706	164	.7249	833	3
58	734	195	.7181	827	2
59	763	225	.7114	821	1
60	.20791	.21256	.47046	.97815	0
N. Cos. N. Cot. N. Tan. N. Sin.					

'	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.20791	.21256	4.7046	.97815	60
1	820	286	.6979	809	59
2	848	316	.6912	803	58
3	877	347	.6845	797	57
4	905	377	.6779	791	56
5	.20933	.21408	4.6712	.97784	55
6	962	438	.6646	778	54
7	.20990	469	.6580	772	53
8	.21019	499	.6514	766	52
9	047	529	.6448	760	51
10	.21076	.21560	4.6382	.97754	50
11	104	590	.6317	748	49
12	132	621	.6252	742	48
13	161	651	.6187	735	47
14	189	682	.6122	729	46
15	.21218	.21712	4.6057	.97723	45
16	246	743	.5993	717	44
17	275	773	.5928	711	43
18	303	804	.5864	705	42
19	331	834	.5800	698	41
20	.21360	.21864	4.5736	.97692	40
21	388	895	.5673	686	39
22	417	925	.5609	680	38
23	445	956	.5546	673	37
24	474	.21986	.5483	667	36
25	.21502	.22017	4.5420	.97661	35
26	530	047	.5357	653	34
27	559	078	.5294	648	33
28	587	108	.5232	642	32
29	616	139	.5169	636	31
30	.21644	.22169	4.5107	.97630	30
31	672	200	.5045	623	29
32	701	231	.4983	617	28
33	729	261	.4922	611	27
34	758	292	.4860	604	26
35	.21786	.22322	4.4799	.97598	25
36	814	353	.4737	592	24
37	843	383	.4676	585	23
38	871	414	.4615	579	22
39	899	444	.4555	573	21
40	.21928	.22475	4.4494	.97566	20
41	956	505	.4434	560	19
42	.21985	536	.4373	553	18
43	.22013	567	.4313	547	17
44	041	597	.4253	541	16
45	.22070	.22628	4.4194	.97534	15
46	098	658	.4134	528	14
47	126	689	.4075	521	13
48	155	719	.4015	515	12
49	183	750	.3956	508	11
50	.22212	.22781	4.3897	.97502	10
51	240	811	.3838	496	9
52	268	842	.3779	489	8
53	297	872	.3721	483	7
54	325	903	.3662	476	6
55	.22353	.22934	4.3604	.97470	5
56	382	964	.3546	463	4
57	410	.22995	.3488	457	3
58	438	.23026	.3430	450	2
59	467	056	.3372	444	1
60	.22495	.23087	4.3315	.97437	0
N. Cos. N. Cot. N. Tan. N. Sin.					'

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.22495	.23087	4.3315	.97437	60
1	523	117	.3257	430	59
2	552	148	.3200	424	58
3	580	179	.3143	417	57
4	608	209	.3086	411	56
5	.22637	.23240	4.3029	.97404	55
6	665	271	.2972	398	54
7	693	301	.2916	391	53
8	722	332	.2859	384	52
9	750	363	.2803	378	51
10	.22778	.23393	4.2747	.97371	50
11	807	424	.2691	365	49
12	835	455	.2635	358	48
13	863	485	.2580	351	47
14	892	516	.2524	345	46
15	.22920	.23547	4.2468	.97338	45
16	948	578	.2413	331	44
17	.22977	608	.2358	325	43
18	.23005	639	.2303	318	42
19	033	670	.2248	311	41
20	.23062	.23700	4.2193	.97304	40
21	090	731	.2139	298	39
22	118	762	.2084	291	38
23	146	793	.2030	284	37
24	175	823	.1976	278	36
25	.23203	.23854	4.1922	.97271	35
26	231	885	.1868	264	34
27	260	916	.1814	257	33
28	288	946	.1760	251	32
29	316	.23977	.1706	244	31
30	.23345	.24008	4.1653	.97237	30
31	373	039	.1600	230	29
32	401	069	.1547	223	28
33	429	100	.1493	217	27
34	458	131	.1441	210	26
35	.23486	.24162	4.1388	.97203	25
36	514	193	.1335	196	24
37	542	223	.1282	189	23
38	571	254	.1230	182	22
39	599	285	.1178	176	21
40	.23627	.24316	4.1126	.97169	20
41	656	347	.1074	162	19
42	684	377	.1022	155	18
43	712	408	.0970	148	17
44	740	439	.0918	141	16
45	.23769	.24470	4.0867	.97134	15
46	797	501	.0815	127	14
47	825	532	.0764	120	13
48	853	562	.0713	113	12
49	882	593	.0662	106	11
50	.23910	.24624	4.0611	.97100	10
51	938	655	.0560	093	
52	966	686	.0509	086	
53	.23995	717	.0459	079	
54	.24023	747	.0408	072	
55	.24051	.24778	4.0358	.97065	
56	079	809	.0308	058	
57	108	840	.0257	051	
58	136	871	.0207	044	
59	164	902	.0158	037	
60	.24192	.24933	4.0108	.97030	
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.24192	.24933	4.0108	.97030	60
1	220	964	.0058	.023	59
2	249	.24993	4.0009	.015	58
3	277	.25026	3.9959	.008	57
4	305	.056	.9910	.97001	56
5	.24333	.25087	3.9861	.96994	55
6	362	118	.9812	.987	54
7	390	149	.9763	.980	53
8	418	180	.9714	.973	52
9	446	211	.9665	.966	51
10	.24474	.25242	3.9617	.96959	50
11	503	273	.9568	.952	49
12	531	304	.9520	.945	48
13	559	335	.9471	.937	47
14	587	366	.9423	.930	46
15	.24615	.25397	3.9375	.96923	45
16	644	428	.9327	.916	44
17	672	459	.9279	.909	43
18	700	490	.9232	.902	42
19	728	521	.9184	.894	41
20	.24756	.25552	3.9136	.96887	40
21	784	583	.9089	.880	39
22	813	614	.9042	.873	38
23	841	645	.8995	.866	37
24	869	676	.8947	.858	36
25	.24897	.25707	3.8900	.96851	35
26	925	738	.8854	.844	34
27	954	769	.8807	.837	33
28	.24982	800	.8760	.829	32
29	.25010	831	.8714	.822	31
30	.25038	.25862	3.8667	.96815	30
31	066	893	.8621	.807	29
32	094	924	.8575	.800	28
33	122	955	.8528	.793	27
34	151	.25986	.8482	.786	26
35	.25179	.26017	3.8436	.96778	25
36	207	048	.8391	.771	24
37	235	079	.8345	.764	23
38	263	110	.8299	.756	22
39	291	141	.8254	.749	21
40	.25320	.26172	3.8208	.96742	20
41	348	203	.8163	.734	19
42	376	235	.8118	.727	18
43	404	266	.8073	.719	17
44	432	297	.8028	.712	16
45	.25460	.26328	3.7983	.96705	15
46	488	359	.7938	.697	14
47	516	390	.7893	.690	13
48	545	421	.7848	.682	12
49	573	452	.7804	.675	11
50	.25601	.26483	3.7760	.96667	10
51	620	515	.7715	.660	9
		546	.7671	.653	8
		577	.7627	.645	7
		608	.7583	.638	6
		.26639	3.7539	.96630	5
		670	.7495	.623	4
		701	.7451	.615	3
		733	.7408	.608	2
		764	.7364	.600	1
	854				
	.25882	.26795	3.7321	.96593	0
N. Cos. N. Cot. N. Tan. N. Sin.					

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.25882	.26795	3.7321	.96593	60
1	910	826	.7277	.585	59
2	938	857	.7234	.578	58
3	966	888	.7191	.570	57
4	.25994	920	.7148	.562	56
5	.26022	.26951	3.7105	.96555	55
6	050	.26982	.7062	.547	54
7	079	.27013	.7019	.540	53
8	107	044	.6976	.532	52
9	135	076	.6933	.524	51
10	.26163	.27107	3.6891	.96517	50
11	191	138	.6848	.509	49
12	219	169	.6806	.502	48
13	247	201	.6764	.494	47
14	275	232	.6722	.486	46
15	.26305	.27263	3.6680	.96479	45
16	331	294	.6638	.471	44
17	359	326	.6596	.463	43
18	387	357	.6554	.456	42
19	415	388	.6512	.448	41
20	.26443	.27419	3.6470	.96440	40
21	471	451	.6429	.433	39
22	500	482	.6387	.425	38
23	528	513	.6346	.417	37
24	556	545	.6305	.410	36
25	.26584	.27576	3.6264	.96402	35
26	612	607	.6222	.394	34
27	640	638	.6181	.386	33
28	668	670	.6140	.379	32
29	696	701	.6100	.371	31
30	.26724	.27732	3.6059	.96363	30
31	752	764	.6018	.355	29
32	780	795	.5978	.347	28
33	808	826	.5937	.340	27
34	836	858	.5897	.332	26
35	.26864	.27889	3.5856	.96324	25
36	892	921	.5816	.316	24
37	920	952	.5776	.308	23
38	948	.27983	.5736	.301	22
39	.26976	.28015	.5696	.293	21
40	.27004	.28046	3.5656	.96285	20
41	032	077	.5616	.277	19
42	060	109	.5576	.269	18
43	088	140	.5536	.261	17
44	116	172	.5497	.253	16
45	.27144	.28203	3.5457	.96246	15
46	172	234	.5418	.238	14
47	200	266	.5379	.230	13
48	228	297	.5339	.222	12
49	256	329	.5300	.214	11
50	.27284	.28360	3.5261	.96206	10
51	312	391	.5222	.198	9
52	340	423	.5183	.190	8
53	368	454	.5144	.182	7
54	396	486	.5105	.174	6
55	.27424	.28517	3.5067	.96166	5
56	452	549	.5028	.158	4
57	480	580	.4989	.150	3
58	508	612	.4951	.142	2
59	536	643	.4912	.134	1
60	.27564	.28675	3.4874	.96126	0
N. Cos. N. Cot. N. Tan. N. Sin.					

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.27564	.28675	3.4874	.96126	60
1	592	706	.4836	118	59
2	620	738	.4798	110	58
3	648	769	.4760	102	57
4	676	801	.4722	094	56
5	.27704	.28832	3.4684	.96086	55
6	731	864	.4646	078	54
7	759	895	.4608	070	53
8	787	927	.4570	062	52
9	815	958	.4533	054	51
10	.27843	.28990	3.4495	.96046	50
11	871	.29021	.4458	037	49
12	899	053	.4420	029	48
13	927	084	.4383	021	47
14	955	116	.4346	013	46
15	.27983	.29147	3.4308	.96005	45
16	.28011	179	.4271	.95997	44
17	039	210	.4234	989	43
18	067	242	.4197	981	42
19	095	274	.4160	972	41
20	.28123	.29305	3.4124	.95964	40
21	150	337	.4087	956	39
22	178	368	.4050	948	38
23	206	400	.4014	940	37
24	234	432	.3977	931	36
25	.28262	.29463	3.3941	.95923	35
26	290	495	.3904	915	34
27	318	526	.3868	907	33
28	346	558	.3832	898	32
29	374	590	.3796	890	31
30	.28402	.29621	3.3759	.95882	30
31	429	653	.3723	874	29
32	457	685	.3687	865	28
33	485	716	.3652	857	27
34	513	748	.3616	849	26
35	.28541	.29780	3.3580	.95841	25
36	569	811	.3544	832	24
37	597	843	.3509	824	23
38	625	875	.3473	816	22
39	652	906	.3438	807	21
40	.28680	.29938	3.3402	.95799	20
41	708	.29970	.3367	791	19
42	736	.30001	.3332	782	18
43	764	033	.3297	774	17
44	792	065	.3261	766	16
45	.28820	.30097	3.3226	.95757	15
46	847	128	.3191	749	14
47	875	160	.3156	740	13
48	903	192	.3122	732	12
49	931	224	.3087	724	11
50	.28959	.30255	3.3052	.95715	10
51	.28987	287	.3017	707	9
52	.29015	319	.2983	698	8
53	042	351	.2948	690	7
54	070	382	.2914	681	6
55	.29098	.30414	3.2879	.95673	5
56	126	446	.2845	664	4
57	154	478	.2811	656	3
58	182	509	.2777	647	2
59	209	541	.2743	639	1
60	.29237	.30573	3.2709	.95630	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.29237	.30573	3.2709	.95630	60
1	265	605	.2675	622	59
2	293	637	.2641	613	58
3	321	669	.2607	605	57
4	348	700	.2573	596	56
5	.29376	.30732	3.2539	.95588	55
6	404	764	.2506	579	54
7	432	796	.2472	571	53
8	460	828	.2438	562	52
9	487	860	.2405	554	51
10	.29515	.30891	3.2371	.95545	50
11	543	923	.2338	536	49
12	571	955	.2305	528	48
13	599	.30987	.2272	519	47
14	626	.31019	.2238	511	46
15	.29654	.31051	3.2205	.95502	45
16	682	083	.2172	493	44
17	710	115	.2139	485	43
18	737	147	.2106	476	42
19	765	178	.2073	467	41
20	.29793	.31210	3.2041	.95459	40
21	821	242	.2008	450	39
22	849	274	.1975	441	38
23	876	306	.1943	433	37
24	904	338	.1910	424	36
25	.29932	.31370	3.1878	.95415	35
26	960	402	.1845	407	34
27	.29987	434	.1813	398	33
28	.30015	466	.1780	389	32
29	043	498	.1748	380	31
30	.30071	.31530	3.1716	.95372	30
31	098	562	.1684	363	29
32	126	594	.1652	354	28
33	154	626	.1620	345	27
34	182	658	.1588	337	26
35	.30209	.31690	3.1556	.95328	25
36	237	722	.1524	319	24
37	265	754	.1492	310	23
38	292	786	.1460	301	22
39	320	818	.1429	293	21
40	.30348	.31850	3.1397	.95284	20
41	376	882	.1366	275	19
42	403	914	.1334	266	18
43	431	946	.1303	257	17
44	459	.31978	.1271	248	16
45	.30486	.32010	3.1240	.95240	15
46	514	042	.1209	231	14
47	542	074	.1178	222	13
48	570	106	.1146	213	12
49	597	139	.1115	204	11
50	.30625	.32171	3.1084	.95195	10
51	653	203	.1053	186	9
52	680	235	.1022	177	8
53	708	267	.0991	168	7
54	736	299	.0961	159	6
55	.30763	.32331	3.0930	.95150	5
56	791	363	.0899	142	4
57	819	396	.0868	133	3
58	846	428	.0838	124	2
59	874	460	.0807	115	1
60	.30902	.32492	3.0777	.95106	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.30902	.32492	3.0777	.95106	60
1	929	524	.0746	.097	59
2	957	556	.0716	.088	58
3	.30985	588	.0686	.079	57
4	.31012	621	.0655	.070	56
5	.31040	.32653	3.0625	.95061	55
6	068	685	.0595	.052	54
7	095	717	.0565	.043	53
8	123	749	.0535	.033	52
9	151	782	.0505	.024	51
10	.31178	.32814	3.0475	.95015	50
11	206	846	.0445	.95006	49
12	233	878	.0415	.94997	48
13	261	911	.0385	.9488	47
14	289	943	.0356	.9479	46
15	.31316	.32975	3.0326	.94970	45
16	344	.33007	.0296	.961	44
17	372	040	.0267	.952	43
18	399	072	.0237	.943	42
19	427	104	.0208	.933	41
20	.31454	.33136	3.0178	.94924	40
21	482	109	.0149	.915	39
22	510	201	.0120	.906	38
23	537	233	.0090	.897	37
24	565	266	.0061	.888	36
25	.31593	.33298	3.0032	.94878	35
26	620	330	3.0003	.869	34
27	648	363	2.9974	.860	33
28	675	395	.9945	.851	32
29	703	427	.9916	.842	31
30	.31730	.33460	2.9887	.94832	30
31	758	492	.9858	.823	29
32	786	524	.9829	.814	28
33	813	557	.9800	.805	27
34	841	589	.9772	.795	26
35	.31868	.33621	2.9743	.94786	25
36	896	654	.9714	.777	24
37	923	686	.9686	.768	23
38	951	718	.9657	.758	22
39	.31979	751	.9629	.749	21
40	.32006	.33783	2.9600	.94740	20
41	034	816	.9572	.730	19
42	061	848	.9544	.721	18
43	089	881	.9515	.712	17
44	116	913	.9487	.702	16
45	.32144	.33945	2.9459	.94693	15
46	171	.33978	.9431	.684	14
47	199	.34010	.9403	.674	13
48	227	043	.9375	.665	12
49	254	075	.9347	.656	11
50	.32282	.34108	2.9319	.94646	10
51	309	140	.9291	.637	9
52	337	173	.9263	.627	8
53	364	205	.9235	.618	7
54	392	238	.9208	.609	6
55	.32419	.34270	2.9180	.94599	5
56	447	303	.9152	.590	4
57	474	335	.9125	.580	3
58	502	368	.9097	.571	2
59	529	400	.9070	.561	1
60	.32557	.34433	2.9042	.94552	0
N. Cos.	N. Cot.	N. Tan.	N. Sin.		

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.32557	.34433	2.9042	.94552	60
1	584	465	.9015	.542	59
2	612	498	.8987	.533	58
3	639	530	.8960	.523	57
4	667	563	.8933	.514	56
5	.32694	.34596	2.8905	.94504	55
6	722	628	.8878	.495	54
7	749	661	.8851	.485	53
8	777	693	.8824	.476	52
9	804	726	.8797	.466	51
10	.32832	.34758	2.8770	.94457	50
11	859	791	.8743	.447	49
12	887	824	.8716	.438	48
13	914	856	.8689	.428	47
14	942	889	.8662	.418	46
15	.32969	.34922	2.8636	.94409	45
16	.32997	954	.8609	.399	44
17	.33024	.34987	.8582	.390	43
18	051	.35020	.8556	.380	42
19	079	052	.8529	.370	41
20	.33106	.35085	2.8502	.94361	40
21	134	118	.8476	.351	39
22	161	150	.8449	.342	38
23	189	183	.8423	.332	37
24	216	216	.8397	.322	36
25	.33244	.35248	2.8370	.94313	35
26	271	281	.8344	.303	34
27	298	314	.8318	.293	33
28	326	346	.8291	.284	32
29	353	379	.8265	.274	31
30	.33381	.35412	2.8239	.94264	30
31	408	445	.8213	.254	29
32	436	477	.8187	.245	28
33	463	510	.8161	.235	27
34	490	543	.8135	.225	26
35	.33518	.35576	2.8109	.94215	25
36	545	608	.8083	.206	24
37	573	641	.8057	.196	23
38	600	674	.8032	.186	22
39	627	707	.8006	.176	21
40	.33655	.35740	2.7980	.94167	20
41	682	772	.7955	.157	19
42	710	805	.7929	.147	18
43	737	838	.7903	.137	17
44	764	871	.7878	.127	16
45	.33792	.35904	2.7852	.94118	15
46	819	937	.7827	.108	14
47	846	.35969	.7801	.098	13
48	874	.36002	.7776	.088	12
49	901	035	.7751	.078	11
50	.33929	.36068	2.7725	.94068	10
51	956	101	.7700	.058	9
52	.33983	134	.7675	.049	8
53	.34011	167	.7650	.039	7
54	038	199	.7625	.029	6
55	.34065	.36232	2.7600	.94019	5
56	093	265	.7575	.94009	4
57	120	298	.7550	.93999	3
58	147	331	.7525	.93989	2
59	175	364	.7500	.93979	1
60	.34202	.36397	2.7475	.93969	0
N. Cos.	N. Cot.	N. Tan.	N. Sin.		

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.34202	.36397	2.7475	.93969	60
1	229	430	.7450	959	59
2	257	463	.7425	949	58
3	284	496	.7400	939	57
4	311	529	.7376	929	56
5	.34339	.36562	2.7351	.93919	55
6	366	595	.7326	909	54
7	393	628	.7302	899	53
8	421	661	.7277	889	52
9	448	694	.7253	879	51
10	.34475	.36727	2.7228	.93869	50
11	503	760	.7204	859	49
12	530	793	.7179	849	48
13	557	826	.7155	839	47
14	584	859	.7130	829	46
15	.34612	.36892	2.7106	.93819	45
16	639	925	.7082	809	44
17	666	958	.7058	799	43
18	694	.36991	.7034	789	42
19	721	.37024	.7009	779	41
20	.34748	.37057	2.6985	.93769	40
21	775	090	.6961	759	39
22	803	123	.6937	748	38
23	830	157	.6913	738	37
24	857	190	.6889	728	36
25	.34884	.37223	2.6865	.93718	35
26	912	256	.6841	708	34
27	939	289	.6818	698	33
28	966	322	.6794	688	32
29	.34993	355	.6770	677	31
30	.35021	.37388	2.6746	.93667	30
31	048	422	.6723	657	29
32	075	455	.6699	647	28
33	102	488	.6675	637	27
34	130	521	.6652	626	26
35	.35157	.37554	2.6628	.93616	25
36	184	588	.6605	606	24
37	211	621	.6581	596	23
38	239	654	.6558	585	22
39	266	687	.6534	575	21
40	.35293	.37720	2.6511	.93565	20
41	320	754	.6488	555	19
42	347	787	.6464	544	18
43	375	820	.6441	534	17
44	402	853	.6418	524	16
45	.35429	.37887	2.6395	.93514	15
46	456	920	.6371	503	14
47	484	953	.6348	493	13
48	511	.37986	.6325	483	12
49	538	.38020	.6302	472	11
50	.35565	.38053	2.6279	.93462	10
51	592	086	.6256	452	9
52	619	120	.6233	441	8
53	647	153	.6210	431	7
54	674	186	.6187	420	6
55	.35701	.38220	2.6165	.93410	5
56	728	253	.6142	400	4
57	755	286	.6119	389	3
58	782	320	.6096	379	2
59	810	353	.6074	368	1
60	.35837	.38386	2.6051	.93358	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.35837	.38386	2.6051	.93358	60
1	864	420	.6028	348	59
2	891	453	.6006	337	58
3	918	487	.5983	327	57
4	945	520	.5961	316	56
5	.35973	.38553	2.5938	.93306	55
6	.36000	587	.5916	295	54
7	027	620	.5893	285	53
8	054	654	.5871	274	52
9	081	687	.5848	264	51
10	.36108	.38721	2.5826	.93253	50
11	135	754	.5804	243	49
12	162	787	.5782	232	48
13	190	821	.5759	222	47
14	217	854	.5737	211	46
15	.36244	.38888	2.5715	.93201	45
16	271	921	.5693	190	44
17	298	953	.5671	180	43
18	325	.38988	.5649	169	42
19	352	.39022	.5627	159	41
20	.36379	.39055	2.5605	.93148	40
21	406	089	.5583	137	39
22	434	122	.5561	127	38
23	461	156	.5539	116	37
24	488	190	.5517	106	36
25	.36515	.39223	2.5495	.93095	35
26	542	257	.5473	084	34
27	569	290	.5452	074	33
28	596	324	.5430	063	32
29	623	357	.5408	052	31
30	.36650	.39391	2.5386	.93042	30
31	677	425	.5365	031	29
32	704	458	.5343	020	28
33	731	492	.5322	.93010	27
34	758	526	.5300	.92999	26
35	.36785	.39559	2.5279	.92988	25
36	812	593	.5257	978	24
37	839	626	.5236	967	23
38	867	660	.5214	956	22
39	894	694	.5193	945	21
40	.36921	.39727	2.5172	.92935	20
41	948	761	.5150	924	19
42	.36975	795	.5129	913	18
43	.37002	829	.5108	902	17
44	029	862	.5086	892	16
45	.37056	.39896	2.5065	.92881	15
46	083	930	.5044	870	14
47	110	963	.5023	859	13
48	137	.39997	.5002	849	12
49	164	.40031	.4981	838	11
50	.37191	.40065	2.4960	.92827	10
51	218	098	.4939	816	9
52	245	132	.4918	805	8
53	272	166	.4897	794	7
54	299	200	.4876	784	6
55	.37326	.40234	2.4855	.92773	5
56	353	267	.4834	762	4
57	380	301	.4813	751	3
58	407	335	.4792	740	2
59	434	369	.4772	729	1
60	.37461	.40403	2.4751	.92718	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.37461	.40403	2.4751	.92718	60
1	488	436	.4730	707	59
2	515	470	.4709	697	58
3	542	504	.4689	686	57
4	569	538	.4668	675	56
5	.37595	.40572	2.4648	.92664	55
6	622	606	.4627	653	54
7	649	640	.4606	642	53
8	676	674	.4586	631	52
9	703	707	.4566	620	51
10	.37730	.40741	2.4545	.92609	50
11	757	775	.4525	598	49
12	784	809	.4504	587	48
13	811	843	.4484	576	47
14	838	877	.4464	565	46
15	.37865	.40911	2.4443	.92554	45
16	892	945	.4423	543	44
17	919	.40979	.4403	532	43
18	946	.41013	.4383	521	42
19	973	047	.4362	510	41
20	.37999	.41081	2.4342	.92499	40
21	.38026	115	.4322	488	39
22	053	149	.4302	477	38
23	080	183	.4282	466	37
24	107	217	.4262	455	36
25	.38134	.41251	2.4242	.92444	35
26	161	285	.4222	432	34
27	188	319	.4202	421	33
28	215	353	.4182	410	32
29	241	387	.4162	399	31
30	.38268	.41421	2.4142	.92388	30
31	295	455	.4122	377	29
32	322	490	.4102	366	28
33	349	524	.4083	355	27
34	376	558	.4063	343	26
35	.38403	.41592	2.4043	.92332	25
36	430	626	.4023	321	24
37	456	660	.4004	310	23
38	483	694	.3984	299	22
39	510	728	.3964	287	21
40	.38537	.41763	2.3945	.92276	20
41	564	797	.3925	265	19
42	591	831	.3906	254	18
43	617	865	.3886	243	17
44	644	899	.3867	231	16
45	.38671	.41933	2.3847	.92220	15
46	698	.41968	.3828	209	14
47	725	.42002	.3808	198	13
48	752	036	.3789	186	12
49	778	070	.3770	175	11
50	.38805	.42105	2.3750	.92164	10
51	832	139	.3731	152	9
52	859	173	.3712	141	8
53	886	207	.3693	130	7
54	912	242	.3673	119	6
55	.38939	.42276	2.3654	.92107	5
56	966	310	.3635	096	4
57	.38993	345	.3616	085	3
58	.39020	379	.3597	073	2
59	046	413	.3578	062	1
60	.39073	.42447	2.3559	.92050	0
N. Cos. N. Cot. N. Tan. N. Sin.					

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.39073	.42447	2.3559	.92050	60
1	100	482	.3539	939	59
2	127	516	.3520	928	58
3	153	551	.3501	916	57
4	180	585	.3483	.92005	56
5	.39207	.42619	2.3464	.91994	55
6	234	654	.3445	982	54
7	260	688	.3426	971	53
8	287	722	.3407	959	52
9	314	757	.3388	948	51
10	.39341	.42791	2.3369	.91936	50
11	367	826	.3351	925	49
12	394	860	.3332	914	48
13	421	894	.3313	902	47
14	448	929	.3294	891	46
15	.39474	.42963	2.3276	.91879	45
16	501	.42998	.3257	868	44
17	528	.43032	.3238	856	43
18	555	067	.3220	845	42
19	581	101	.3201	833	41
20	.39608	.43136	2.3183	.91822	40
21	635	170	.3164	810	39
22	661	205	.3146	799	38
23	688	239	.3127	787	37
24	715	274	.3109	775	36
25	.39741	.43308	2.3090	.91764	35
26	768	343	.3072	752	34
27	795	378	.3053	741	33
28	822	412	.3035	729	32
29	848	447	.3017	718	31
30	.39875	.43481	2.2998	.91706	30
31	902	516	.2980	694	29
32	928	550	.2962	683	28
33	955	585	.2944	671	27
34	.39982	620	.2925	660	26
35	.40008	.43654	2.2907	.91648	25
36	035	689	.2889	636	24
37	062	724	.2871	625	23
38	088	758	.2853	613	22
39	115	793	.2835	601	21
40	.40141	.43828	2.2817	.91590	20
41	168	862	.2799	578	19
42	195	897	.2781	566	18
43	221	932	.2763	555	17
44	248	.43966	.2745	543	16
45	.40275	.44001	2.2727	.91531	15
46	301	036	.2709	519	14
47	328	071	.2691	508	13
48	355	105	.2673	496	12
49	381	140	.2655	484	11
50	.40408	.44175	2.2637	.91472	10
51	434	210	.2620	461	9
52	461	244	.2602	449	8
53	488	279	.2584	437	7
54	514	314	.2566	425	6
55	.40541	.44349	2.2549	.91414	5
56	567	384	.2531	402	4
57	594	418	.2513	390	3
58	621	453	.2496	378	2
59	647	488	.2478	366	1
60	.40674	.44523	2.2460	.91355	0
N. Cos. N. Cot. N. Tan. N. Sin.					

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.40674	.44523	2.2460	.91355	60
1	700	558	.2443	343	59
2	727	593	.2425	331	58
3	753	627	.2408	319	57
4	780	662	.2390	307	56
5	.40806	.44697	2.2373	.91295	55
6	833	732	.2355	283	54
7	860	767	.2338	272	53
8	886	802	.2320	260	52
9	913	837	.2303	248	51
10	.40939	.44872	2.2286	.91236	50
11	966	907	.2268	224	49
12	.40992	942	.2251	212	48
13	.41019	.44977	.2234	200	47
14	045	.45012	.2216	188	46
15	.41072	.45047	2.2199	.91176	45
16	098	082	.2182	164	44
17	125	117	.2165	152	43
18	151	152	.2148	140	42
19	178	187	.2130	128	41
20	.41204	.45222	2.2113	.91116	40
21	231	257	.2096	104	39
22	257	292	.2079	92	38
23	284	327	.2062	080	37
24	310	362	.2045	068	36
25	.41337	.45397	2.2028	.91056	35
26	363	432	.2011	044	34
27	390	467	.1994	032	33
28	416	502	.1977	020	32
29	443	538	.1960	.91008	31
30	.41469	.45573	2.1943	.90996	30
31	496	608	.1926	984	29
32	522	643	.1909	972	28
33	549	678	.1892	960	27
34	575	713	.1876	948	26
35	.41602	.45748	2.1859	.90936	25
36	628	784	.1842	924	24
37	655	819	.1825	911	23
38	681	854	.1808	899	22
39	707	889	.1792	887	21
40	.41734	.45924	2.1775	.90875	20
41	760	960	.1758	863	19
42	787	.45995	.1742	851	18
43	813	.46030	.1725	839	17
44	840	065	.1708	826	16
45	.41866	.46101	2.1692	.90814	15
46	892	136	.1675	802	14
47	919	171	.1659	790	13
48	945	206	.1642	778	12
49	972	242	.1625	766	11
50	.41998	.46277	2.1609	.90753	10
51	.42024	312	.1592	741	9
52	051	348	.1576	729	8
53	077	383	.1560	717	7
54	104	418	.1543	704	6
55	.42130	.46454	2.1527	.90692	5
56	156	489	.1510	680	4
57	183	525	.1494	668	3
58	209	560	.1478	655	2
59	235	595	.1461	643	1
60	.42262	.46631	2.1445	.90631	0
N. Cos.		N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.42262	.46631	2.1445	.90631	60
1	288	666	.1429	618	59
2	315	702	.1413	606	58
3	341	737	.1396	594	57
4	367	772	.1380	582	56
5	.42394	.46808	2.1364	.90569	55
6	420	843	.1348	557	54
7	446	879	.1332	545	53
8	473	914	.1315	532	52
9	499	950	.1299	520	51
10	.42525	.46985	2.1283	.90507	50
11	552	.47021	.1267	495	49
12	578	056	.1251	483	48
13	604	092	.1235	470	47
14	631	128	.1219	458	46
15	.42657	.47163	2.1203	.90446	45
16	683	199	.1187	433	44
17	709	234	.1171	421	43
18	736	270	.1155	408	42
19	762	305	.1139	396	41
20	.42788	.47341	2.1123	.90383	40
21	815	377	.1107	371	39
22	841	412	.1092	358	38
23	867	448	.1076	346	37
24	894	483	.1060	334	36
25	.42920	.47519	2.1044	.90321	35
26	946	555	.1028	309	34
27	972	590	.1013	296	33
28	.42999	626	.0997	284	32
29	.43025	662	.0981	271	31
30	.43051	.47698	2.0965	.90259	30
31	077	733	.0950	246	29
32	104	769	.0934	233	28
33	130	805	.0918	221	27
34	156	840	.0903	208	26
35	.43182	.47876	2.0887	.90196	25
36	209	912	.0872	183	24
37	235	948	.0856	171	23
38	261	.47984	.0840	158	22
39	287	.48019	.0825	146	21
40	.43313	.48055	2.0809	.90133	20
41	340	091	.0794	120	19
42	366	127	.0778	108	18
43	392	163	.0763	095	17
44	418	198	.0748	082	16
45	.43445	.48234	2.0732	.90070	15
46	471	270	.0717	057	14
47	497	306	.0701	045	13
48	523	342	.0686	032	12
49	549	378	.0671	019	11
50	.43575	.48414	2.0655	.90007	10
51	602	.450	.0640	.89994	9
52	628	486	.0625	981	8
53	654	521	.0609	968	7
54	680	557	.0594	956	6
55	.43706	.48593	2.0579	.89943	5
56	733	629	.0564	930	4
57	759	665	.0549	918	3
58	785	701	.0533	905	2
59	811	737	.0518	892	1
60	.43837	.48773	2.0503	.89879	0
N. Cos.		N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.43837	.48773	2.0503	.89879	60
1	863	809	.0488	867	59
2	889	845	.0473	854	58
3	916	881	.0458	841	57
4	942	917	.0443	828	56
5	.43968	.48953	2.0428	.89816	55
6	.43994	.48989	.0413	803	54
7	.44020	.49026	.0398	790	53
8	046	062	.0383	777	52
9	072	098	.0368	764	51
10	.44098	.49134	2.0353	.89752	50
11	124	170	.0338	739	49
12	151	206	.0323	726	48
13	177	242	.0308	713	47
14	203	278	.0293	700	46
15	.44229	.49315	2.0278	.89687	45
16	255	351	.0263	674	44
17	281	387	.0248	662	43
18	307	423	.0233	649	42
19	333	459	.0219	636	41
20	.44359	.49495	2.0204	.89623	40
21	385	532	.0189	610	39
22	411	568	.0174	597	38
23	437	604	.0160	584	37
24	464	640	.0145	571	36
25	.44490	.49677	2.0130	.89558	35
26	516	713	.0115	545	34
27	542	749	.0101	532	33
28	568	786	.0086	519	32
29	594	822	.0072	506	31
30	.44620	.49858	2.0057	.89493	30
31	646	894	.0042	480	29
32	672	931	.0028	467	28
33	698	.49967	2.0013	454	27
34	724	.50004	1.9999	441	26
35	.44750	.50040	1.9984	.89428	25
36	776	076	.9970	415	24
37	802	113	.9955	402	23
38	828	149	.9941	389	22
39	854	185	.9926	376	21
40	.44880	.50222	1.9912	.89363	20
41	906	258	.9897	350	19
42	932	295	.9883	337	18
43	958	331	.9868	324	17
44	.44984	368	.9854	311	16
45	.45010	.50404	1.9840	.89298	15
46	036	441	.9825	285	14
47	062	477	.9811	272	13
48	088	514	.9797	259	12
49	114	550	.9782	245	11
50	.45140	.50587	1.9768	.89232	10
51	166	623	.9754	219	9
52	192	660	.9740	206	8
53	218	696	.9725	193	7
54	243	733	.9711	180	6
55	.45269	.50769	1.9697	.89167	5
56	295	806	.9683	153	4
57	321	843	.9669	140	3
58	347	879	.9654	127	2
59	373	916	.9640	114	1
60	.45399	.50953	1.9626	.89101	0
N. Cos. N. Cot. N. Tan. N. Sin.					

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.45399	.50953	1.9626	.89101	60
1	425	.50989	.9612	087	59
2	451	.51026	.9598	074	58
3	477	063	.9584	061	57
4	503	099	.9570	048	56
5	.45529	.51136	1.9556	.89035	55
6	554	173	.9542	021	54
7	580	209	.9528	008	53
8	606	246	.9514	.88995	52
9	632	283	.9500	981	51
10	.45658	.51319	1.9486	.88968	50
11	684	356	.9472	955	49
12	710	393	.9458	942	48
13	736	430	.9444	928	47
14	762	467	.9430	915	46
15	.45787	.51503	1.9416	.88902	45
16	813	540	.9402	888	44
17	839	577	.9388	875	43
18	865	614	.9375	862	42
19	891	651	.9361	848	41
20	.45917	.51688	1.9347	.88835	40
21	942	724	.9333	822	39
22	968	761	.9319	808	38
23	.45994	798	.9306	795	37
24	.46020	835	.9292	782	36
25	.46046	.51872	1.9278	.88768	35
26	072	909	.9265	755	34
27	097	946	.9251	741	33
28	123	.51983	.9237	728	32
29	149	.52020	.9223	715	31
30	.46175	.52057	1.9210	.88701	30
31	201	094	.9196	688	29
32	226	131	.9183	674	28
33	252	168	.9169	661	27
34	278	205	.9155	647	26
35	.46304	.52242	1.9142	.88634	25
36	330	279	.9128	620	24
37	355	316	.9115	607	23
38	381	353	.9101	593	22
39	407	390	.9088	580	21
40	.46433	.52427	1.9074	.88566	20
41	458	464	.9061	553	19
42	484	501	.9047	539	18
43	510	538	.9034	526	17
44	536	575	.9020	512	16
45	.46561	.52613	1.9007	.88499	15
46	587	650	.9093	485	14
47	613	687	.9080	472	13
48	639	724	.9067	458	12
49	664	761	.9053	445	11
50	.46690	.52798	1.8940	.88431	10
51	716	836	.8927	417	9
52	742	873	.8913	404	8
53	767	910	.8900	390	7
54	793	947	.8887	377	6
55	.46819	.52985	1.8873	.88363	5
56	844	.53022	.8860	349	4
57	870	059	.8847	336	3
58	896	096	.8834	322	2
59	921	134	.8820	308	1
60	.46947	.53171	1.8807	.88295	0
N. Cos. N. Cot. N. Tan. N. Sin.					

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.46947	.53171	1.8807	.88295	60
1	973	208	.8794	281	59
2	.46999	246	.8781	267	58
3	.47024	283	.8768	254	57
4	050	320	.8755	240	56
5	.47076	.53358	1.8741	.88226	55
6	101	395	.8728	213	54
7	127	432	.8715	199	53
8	153	470	.8702	185	52
9	178	507	.8689	172	51
10	.47204	.53545	1.8676	.88158	50
11	229	582	.8663	144	49
12	255	620	.8650	130	48
13	281	657	.8637	117	47
14	306	694	.8624	103	46
15	.47332	.53732	1.8611	.88089	45
16	358	769	.8598	075	44
17	383	807	.8585	062	43
18	409	844	.8572	048	42
19	434	882	.8559	034	41
20	.47460	.53920	1.8546	.88020	40
21	486	957	.8533	.88006	39
22	511	.53995	.8520	.87993	38
23	537	.54032	.8507	979	37
24	562	070	.8495	965	36
25	.47588	.54107	1.8482	.87951	35
26	614	145	.8469	937	34
27	639	183	.8456	923	33
28	665	220	.8443	909	32
29	690	258	.8430	896	31
30	.47716	.54296	1.8418	.87882	30
31	741	383	.8405	868	29
32	767	371	.8392	854	28
33	793	409	.8379	840	27
34	818	446	.8367	826	26
35	.47844	.54484	1.8354	.87812	25
36	869	522	.8341	798	24
37	895	560	.8329	784	23
38	920	597	.8316	770	22
39	946	635	.8303	756	21
40	.47971	.54673	1.8291	.87743	20
41	.47997	711	.8278	729	19
42	.48022	748	.8265	715	18
43	048	786	.8253	701	17
44	073	824	.8240	687	16
45	.48099	.54862	1.8228	.87673	15
46	124	900	.8215	659	14
47	150	938	.8202	645	13
48	175	.54975	.8190	631	12
49	201	.55013	.8177	617	11
50	.48226	.55051	1.8165	.87603	10
51	252	089	.8152	589	9
52	277	127	.8140	575	8
53	303	165	.8127	561	7
54	328	203	.8115	546	6
55	.48354	.55241	1.8103	.87532	5
56	379	279	.8090	518	4
57	405	317	.8078	504	3
58	430	355	.8065	490	2
59	456	393	.8053	476	1
60	.48481	.55431	1.8040	.87462	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.48481	.55431	1.8040	.87462	60
1	506	469	.8028	448	59
2	532	507	.8016	434	58
3	557	545	.8003	420	57
4	583	583	.7991	406	56
5	.48608	.55621	1.7979	.87391	55
6	634	659	.7966	377	54
7	659	697	.7954	363	53
8	684	736	.7942	349	52
9	710	774	.7930	335	51
10	.48735	.55812	1.7917	.87321	50
11	761	850	.7905	306	49
12	786	888	.7893	292	48
13	811	926	.7881	278	47
14	837	.55964	.7868	264	46
15	.48862	.56003	1.7856	.87250	45
16	888	041	.7844	235	44
17	913	079	.7832	221	43
18	938	117	.7820	207	42
19	964	156	.7808	193	41
20	.48989	.56194	1.7796	.87178	40
21	.49014	232	.7783	164	39
22	040	270	.7771	150	38
23	065	309	.7759	136	37
24	090	347	.7747	121	36
25	.49116	.56385	1.7735	.87107	35
26	141	424	.7723	093	34
27	166	462	.7711	079	33
28	192	501	.7699	064	32
29	217	539	.7687	050	31
30	.49242	.56577	1.7675	.87036	30
31	268	616	.7663	021	29
32	293	654	.7651	.87007	28
33	318	693	.7639	.86993	27
34	344	731	.7627	978	26
35	.49369	.56769	1.7615	.86964	25
36	394	808	.7603	949	24
37	419	846	.7591	935	23
38	445	885	.7579	921	22
39	470	923	.7567	906	21
40	.49495	.56962	1.7556	.86892	20
41	521	.57000	.7544	878	19
42	546	039	.7532	863	18
43	571	078	.7520	849	17
44	596	116	.7508	834	16
45	.49622	.57155	1.7496	.86820	15
46	647	193	.7485	805	14
47	672	232	.7473	791	13
48	697	271	.7461	777	12
49	723	309	.7449	762	11
50	.49748	.57348	1.7437	.86748	10
51	773	386	.7426	733	9
52	798	425	.7414	719	
53	824	464	.7402	704	
54	849	503	.7391	690	
55	.49874	.57541	1.7379	.86675	
56	899	580	.7367	661	
57	924	619	.7355	646	
58	950	657	.7344	632	
59	.49975	696	.7332	617	
60	.50000	.57735	1.7321	.86603	
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.50000	.57735	1.7321	.86603	60
1	.025	.774	.7309	.588	59
2	.050	.813	.7297	.573	58
3	.076	.851	.7286	.559	57
4	.101	.890	.7274	.544	56
5	.50126	.57929	1.7262	.86530	55
6	.151	.57968	.7251	.515	54
7	.176	.58007	.7239	.501	53
8	.201	.046	.7228	.486	52
9	.227	.085	.7216	.471	51
10	.50252	.58124	1.7205	.86457	50
11	.277	.162	.7193	.442	49
12	.302	.201	.7182	.427	48
13	.327	.240	.7170	.413	47
14	.352	.279	.7159	.398	46
15	.50377	.58318	1.7147	.86384	45
16	.403	.357	.7136	.369	44
17	.428	.396	.7124	.354	43
18	.453	.435	.7113	.340	42
19	.478	.474	.7102	.325	41
20	.50503	.58513	1.7090	.86310	40
21	.528	.552	.7079	.295	39
22	.553	.591	.7067	.281	38
23	.578	.631	.7056	.266	37
24	.603	.670	.7045	.251	36
25	.50628	.58709	1.7033	.86237	35
26	.654	.748	.7022	.222	34
27	.679	.787	.7011	.207	33
28	.704	.826	.6999	.192	32
29	.729	.865	.6988	.178	31
30	.50754	.58905	1.6977	.86163	30
31	.779	.944	.6965	.148	29
32	.804	.58983	.6954	.133	28
33	.829	.59022	.6943	.119	27
34	.854	.061	.6932	.104	26
35	.50879	.59101	1.6920	.86089	25
36	.904	.140	.6909	.074	24
37	.929	.179	.6898	.059	23
38	.954	.218	.6887	.045	22
39	.50979	.258	.6875	.030	21
40	.51004	.59297	1.6864	.86015	20
41	.029	.336	.6853	.86000	19
42	.054	.376	.6842	.85985	18
43	.079	.415	.6831	.970	17
44	.104	.454	.6820	.956	16
45	.51129	.59494	1.6808	.85941	15
46	.154	.533	.6797	.926	14
47	.179	.573	.6786	.911	13
48	.204	.612	.6775	.896	12
49	.229	.651	.6764	.881	11
50	.51254	.59691	1.6753	.85866	10
51	.279	.730	.6742	.851	9
52	.304	.770	.6731	.836	8
53	.329	.809	.6720	.821	7
54	.354	.849	.6709	.806	6
55	.51379	.59888	1.6698	.85792	5
56	.404	.928	.6687	.777	4
57	.429	.59967	.6676	.762	3
58	.454	.60007	.6665	.747	2
59	.479	.046	.6654	.732	1
60	.51504	.60086	1.6643	.85717	0
N. Cos.		N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.51504	.60086	1.6643	.85717	60
1	.529	.126	.6632	.702	59
2	.554	.165	.6621	.687	58
3	.579	.205	.6610	.672	57
4	.604	.245	.6599	.657	56
5	.51628	.60284	1.6588	.85642	55
6	.653	.324	.6577	.627	54
7	.678	.364	.6566	.612	53
8	.703	.403	.6555	.597	52
9	.728	.443	.6545	.582	51
10	.51753	.60483	1.6534	.85567	50
11	.778	.522	.6523	.551	49
12	.803	.562	.6512	.536	48
13	.828	.602	.6501	.521	47
14	.852	.642	.6490	.506	46
15	.51877	.60681	1.6479	.85491	45
16	.902	.721	.6469	.476	44
17	.927	.761	.6458	.461	43
18	.952	.801	.6447	.446	42
19	.51977	.841	.6436	.431	41
20	.52002	.60881	1.6426	.85416	40
21	.026	.921	.6415	.401	39
22	.051	.60960	.6404	.385	38
23	.076	.61000	.6393	.370	37
24	.101	.040	.6383	.355	36
25	.52126	.61080	1.6372	.85340	35
26	.151	.120	.6361	.325	34
27	.175	.160	.6351	.310	33
28	.200	.200	.6340	.294	32
29	.225	.240	.6329	.279	31
30	.52250	.61280	1.6319	.85264	30
31	.275	.320	.6308	.249	29
32	.299	.360	.6297	.234	28
33	.324	.400	.6287	.218	27
34	.349	.440	.6276	.203	26
35	.52374	.61480	1.6265	.85188	25
36	.399	.520	.6255	.173	24
37	.423	.561	.6244	.157	23
38	.448	.601	.6234	.142	22
39	.473	.641	.6223	.127	21
40	.52498	.61681	1.6212	.85112	20
41	.522	.721	.6202	.096	19
42	.547	.761	.6191	.081	18
43	.572	.801	.6181	.066	17
44	.597	.842	.6170	.051	16
45	.52621	.61882	1.6160	.85035	15
46	.646	.922	.6149	.020	14
47	.671	.61962	.6139	.85005	13
48	.696	.62003	.6128	.84989	12
49	.720	.043	.6118	.974	11
50	.52745	.62083	1.6107	.84959	10
51	.770	.124	.6097	.943	9
52	.794	.164	.6087	.928	8
53	.819	.204	.6076	.913	7
54	.844	.245	.6066	.897	6
55	.52869	.62285	1.6055	.84882	5
56	.893	.325	.6045	.866	4
57	.918	.366	.6034	.851	3
58	.943	.406	.6024	.836	2
59	.967	.446	.6014	.820	1
60	.52992	.62487	1.6003	.84805	0
N. Cos.		N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.52992	.62487	1.6003	.84805	60
1	.53017	527	.5993	789	59
2	041	568	.5983	774	58
3	066	608	.5972	759	57
4	091	649	.5962	743	56
5	.53115	.62689	1.5952	.84728	55
6	140	730	.5941	712	54
7	164	770	.5931	697	53
8	189	811	.5921	681	52
9	214	852	.5911	666	51
10	.53238	.62892	1.5900	.84650	50
11	263	933	.5890	635	49
12	288	.62973	.5880	619	48
13	312	.63014	.5869	604	47
14	337	055	.5859	588	46
15	.53361	.63095	1.5849	.84573	45
16	386	136	.5839	557	44
17	411	177	.5829	542	43
18	435	217	.5818	526	42
19	460	258	.5808	511	41
20	.53484	.63299	1.5798	.84495	40
21	509	340	.5788	480	39
22	534	380	.5778	464	38
23	558	421	.5768	448	37
24	583	462	.5757	433	36
25	.53607	.63503	1.5747	.84417	35
26	632	544	.5737	402	34
27	656	584	.5727	386	33
28	681	625	.5717	370	32
29	705	666	.5707	355	31
30	.53730	.63707	1.5697	.84339	30
31	754	748	.5687	324	29
32	779	789	.5677	308	28
33	804	830	.5667	292	27
34	828	871	.5657	277	26
35	.53853	.63912	1.5647	.84261	25
36	877	953	.5637	245	24
37	902	.63994	.5627	230	23
38	926	.64035	.5617	214	22
39	951	076	.5607	198	21
40	.53975	.64117	1.5597	.84182	20
41	.54000	158	.5587	167	19
42	024	199	.5577	151	18
43	049	240	.5567	135	17
44	073	281	.5557	120	16
45	.54097	.64322	1.5547	.84104	15
46	122	363	.5537	088	14
47	146	404	.5527	072	13
48	171	446	.5517	057	12
49	195	487	.5507	041	11
50	.54220	.64528	1.5497	.84025	10
51	244	569	.5487	.84009	9
52	269	610	.5477	.83994	8
53	293	652	.5468	978	7
54	317	693	.5458	962	6
55	.54342	.64734	1.5448	.83946	5
56	366	775	.5438	930	4
57	391	817	.5428	915	3
58	415	858	.5418	899	2
59	440	899	.5408	883	1
60	.54464	.64941	1.5399	.83867	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.54464	.64941	1.5399	.83867	60
1	488	.64982	.5389	851	59
2	513	.65024	.5379	835	58
3	537	065	.5369	819	57
4	561	106	.5359	804	56
5	.54586	.65148	1.5350	.83788	55
6	610	189	.5340	772	54
7	635	231	.5330	756	53
8	659	272	.5320	740	52
9	683	314	.5311	724	51
10	.54708	.65355	1.5301	.83708	50
11	732	397	.5291	692	49
12	756	438	.5282	676	48
13	781	480	.5272	660	47
14	805	521	.5262	645	46
15	.54829	.65563	1.5253	.83629	45
16	854	604	.5243	613	44
17	878	646	.5233	597	43
18	902	688	.5224	581	42
19	927	729	.5214	565	41
20	.54951	.65771	1.5204	.83549	40
21	975	813	.5195	533	39
22	.54999	854	.5185	517	38
23	.55024	896	.5175	501	37
24	048	938	.5166	485	36
25	.55072	.65980	1.5156	.83469	35
26	097	.66021	.5147	453	34
27	121	063	.5137	437	33
28	145	105	.5127	421	32
29	169	147	.5118	405	31
30	.55194	.66189	1.5108	.83389	30
31	218	230	.5099	373	29
32	242	272	.5089	356	28
33	266	314	.5080	340	27
34	291	356	.5070	324	26
35	.55315	.66398	1.5061	.83308	25
36	339	440	.5051	292	24
37	363	482	.5042	276	23
38	388	524	.5032	260	22
39	412	566	.5023	244	21
40	.55436	.66608	1.5013	.83228	20
41	460	650	.5004	212	19
42	484	692	.4994	195	18
43	509	734	.4985	179	17
44	533	776	.4975	163	16
45	.55557	.66818	1.4966	.83147	15
46	581	860	.4957	131	14
47	605	902	.4947	115	13
48	630	944	.4938	098	12
49	654	.66986	.4928	082	11
50	.55678	.67028	1.4919	.83066	10
51	702	071	.4910	050	9
52	726	113	.4900	034	8
53	750	155	.4891	017	7
54	775	197	.4882	.83001	6
55	.55799	.67239	1.4872	.82985	5
56	823	282	.4863	969	4
57	847	324	.4854	953	3
58	871	366	.4844	936	2
59	895	409	.4835	920	1
60	.55919	.67451	1.4826	.82904	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.55919	.67451	1.4826	.82904	60
1	943	493	.4816	887	59
2	968	536	.4807	871	58
3	.55992	578	.4798	855	57
4	.56016	620	.4788	839	56
5	.56040	.67663	1.4779	.82822	55
6	064	705	.4770	806	54
7	088	748	.4761	790	53
8	112	790	.4751	773	52
9	136	832	.4742	757	51
10	.56160	.67875	1.4733	.82741	50
11	184	917	.4724	724	49
12	208	.67960	.4715	708	48
13	232	.68002	.4705	692	47
14	256	045	.4696	675	46
15	.56280	.68088	1.4687	.82659	45
16	305	130	.4678	643	44
17	329	173	.4669	626	43
18	353	215	.4659	610	42
19	377	258	.4650	593	41
20	.56401	.68301	1.4641	.82577	40
21	425	343	.4632	561	39
22	449	386	.4623	544	38
23	473	429	.4614	528	37
24	497	471	.4605	511	36
25	.56521	.68514	1.4596	.82495	35
26	545	557	.4586	478	34
27	569	600	.4577	462	33
28	593	642	.4568	446	32
29	617	685	.4559	429	31
30	.56641	.68728	1.4550	.82413	30
31	665	771	.4541	396	29
32	689	814	.4532	380	28
33	713	857	.4523	363	27
34	736	900	.4514	347	26
35	.56760	.68942	1.4505	.82330	25
36	784	.68985	.4496	314	24
37	808	.69028	.4487	297	23
38	832	071	.4478	281	22
39	856	114	.4469	264	21
40	.56880	.69157	1.4460	.82248	20
41	904	200	.4451	231	19
42	928	243	.4442	214	18
43	952	286	.4433	198	17
44	.56976	329	.4424	181	16
45	.57000	.69372	1.4415	.82165	15
46	024	416	.4406	148	14
47	047	459	.4397	132	13
48	071	502	.4388	115	12
49	095	545	.4379	098	11
50	.57119	.69588	1.4370	.82082	10
51	143	631	.4361	065	9
52	167	675	.4352	048	8
53	191	718	.4344	032	7
54	215	761	.4335	.82015	6
55	.57238	.69804	1.4326	.81999	5
56	262	847	.4317	982	4
57	286	891	.4308	965	3
58	310	934	.4299	949	2
59	334	.69977	.4290	932	1
60	.57358	.70021	1.4281	.81915	0
N. Cos. N. Cot. N. Tan. N. Sin.					

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.57358	.70021	1.4281	.81915	60
1	381	064	.4273	899	59
2	405	107	.4264	882	58
3	429	151	.4255	865	57
4	453	194	.4246	848	56
5	.57477	.70238	1.4237	.81832	55
6	501	281	.4229	815	54
7	524	325	.4220	798	53
8	548	368	.4211	782	52
9	572	412	.4202	765	51
10	.57596	.70455	1.4193	.81748	50
11	619	499	.4185	731	49
12	643	542	.4176	714	48
13	667	586	.4167	698	47
14	691	629	.4158	681	46
15	.57715	.70673	1.4150	.81664	45
16	738	717	.4141	647	44
17	762	760	.4132	631	43
18	786	804	.4124	614	42
19	810	848	.4115	597	41
20	.57833	.70891	1.4106	.81580	40
21	857	935	.4097	563	39
22	881	.70979	.4089	546	38
23	904	.71023	.4080	530	37
24	928	066	.4071	513	36
25	.57952	.71110	1.4063	.81496	35
26	976	154	.4054	479	34
27	.57999	198	.4045	462	33
28	.58023	242	.4037	445	32
29	047	285	.4028	428	31
30	.58070	.71329	1.4019	.81412	30
31	094	373	.4011	395	29
32	118	417	.4002	378	28
33	141	461	.3994	361	27
34	165	505	.3985	344	26
35	.58189	.71549	1.3976	.81327	25
36	212	593	.3968	310	24
37	236	637	.3959	293	23
38	260	681	.3951	276	22
39	283	725	.3942	259	21
40	.58307	.71769	1.3934	.81242	20
41	330	813	.3925	225	19
42	354	857	.3916	208	18
43	378	901	.3908	191	17
44	401	946	.3899	174	16
45	.58425	.71990	1.3891	.81157	15
46	449	.72034	.3882	140	14
47	472	078	.3874	123	13
48	496	122	.3865	106	12
49	519	167	.3857	089	11
50	.58543	.72211	1.3848	.81072	10
51	567	255	.3840	055	9
52	590	299	.3831	038	8
53	614	344	.3823	021	7
54	637	388	.3814	.81004	6
55	.58661	.72432	1.3806	.80987	5
56	684	477	.3798	970	4
57	708	521	.3789	953	3
58	731	565	.3781	936	2
59	755	610	.3772	919	1
60	.58779	.72654	1.3764	.80902	0
N. Cos. N. Cot. N. Tan. N. Sin.					

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.58779	.72654	1.3764	.80902	60
1	802	699	.3755	885	59
2	826	743	.3747	867	58
3	849	788	.3739	850	57
4	873	832	.3730	833	56
5	.58896	.72877	1.3722	.80816	55
6	920	921	.3713	799	54
7	943	.72966	.3705	782	53
8	967	.73010	.3697	765	52
9	.58990	055	.3688	748	51
10	.59014	.73100	1.3680	.80730	50
11	037	144	.3672	713	49
12	061	189	.3663	696	48
13	084	234	.3655	679	47
14	108	278	.3647	662	46
15	.59131	.73323	1.3638	.80644	45
16	154	368	.3630	627	44
17	178	413	.3622	610	43
18	201	457	.3613	593	42
19	225	502	.3605	576	41
20	.59248	.73547	1.3597	.80558	40
21	272	592	.3588	541	39
22	295	637	.3580	524	38
23	318	681	.3572	507	37
24	342	726	.3564	489	36
25	.59365	.73771	1.3555	.80472	35
26	389	816	.3547	455	34
27	412	861	.3539	438	33
28	436	906	.3531	420	32
29	459	951	.3522	403	31
30	.59482	.73996	1.3514	.80386	30
31	506	.74041	.3506	368	29
32	529	086	.3498	351	28
33	552	131	.3490	334	27
34	576	176	.3481	316	26
35	.59599	.74221	1.3473	.80299	25
36	622	267	.3465	282	24
37	646	312	.3457	264	23
38	669	357	.3449	247	22
39	693	402	.3440	230	21
40	.59716	.74447	1.3432	.80212	20
41	739	492	.3424	195	19
42	763	538	.3416	178	18
43	786	583	.3408	160	17
44	809	628	.3400	143	16
45	.59832	.74674	1.3392	.80122	15
46	856	719	.3384	108	14
47	879	764	.3375	091	13
48	902	810	.3367	073	12
49	926	855	.3359	056	11
50	.59949	.74900	1.3351	.80038	10
51	972	946	.3343	021	9
52	.59995	.74991	.3335	.80003	8
53	.60019	.75037	.3327	.79986	7
54	042	082	.3319	968	6
55	.60065	.75128	1.3311	.79951	5
56	089	173	.3303	934	4
57	112	219	.3295	916	3
58	135	264	.3287	899	2
59	158	310	.3278	881	1
60	.60182	.75355	1.3270	.79864	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.60182	.75355	1.3270	.79864	60
1	205	401	.3262	846	59
2	228	447	.3254	829	58
3	251	492	.3246	811	57
4	274	538	.3238	793	56
5	.60298	.75584	1.3230	.79776	55
6	321	629	.3222	758	54
7	344	675	.3214	741	53
8	367	721	.3206	723	52
9	390	767	.3198	706	51
10	.60414	.75812	1.3190	.79688	50
11	437	858	.3182	671	49
12	460	904	.3175	653	48
13	483	950	.3167	635	47
14	506	.75996	.3159	618	46
15	.60529	.76042	1.3151	.79600	45
16	553	088	.3143	583	44
17	576	134	.3135	565	43
18	599	180	.3127	547	42
19	622	226	.3119	530	41
20	.60645	.76272	1.3111	.79512	40
21	668	318	.3103	494	39
22	691	364	.3095	477	38
23	714	410	.3087	459	37
24	738	456	.3079	441	36
25	.60761	.76502	1.3072	.79424	35
26	784	548	.3064	406	34
27	807	594	.3056	388	33
28	830	640	.3048	371	32
29	853	686	.3040	353	31
30	.60876	.76733	1.3032	.79335	30
31	899	779	.3024	318	29
32	922	825	.3017	300	28
33	945	871	.3009	282	27
34	968	918	.3001	264	26
35	.60991	.76964	1.2993	.79247	25
36	.61015	.77010	.2985	229	24
37	038	057	.2977	211	23
38	061	103	.2970	193	22
39	084	149	.2962	176	21
40	.61107	.77196	1.2954	.79158	20
41	130	242	.2946	140	19
42	153	289	.2938	122	18
43	176	335	.2931	105	17
44	199	382	.2923	087	16
45	.61222	.77428	1.2915	.79069	15
46	245	475	.2907	051	14
47	268	521	.2900	033	13
48	291	568	.2892	.79016	12
49	314	615	.2884	.78998	11
50	.61337	.77661	1.2876	.78980	10
51	360	708	.2869	962	9
52	383	754	.2861	944	8
53	406	801	.2853	926	7
54	429	848	.2846	908	6
55	.61451	.77895	1.2838	.78891	5
56	474	941	.2830	873	4
57	497	.77988	.2822	855	3
58	520	.78035	.2815	837	2
59	543	082	.2807	819	1
60	.61566	.78129	1.2799	.78801	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.61566	.78129	1.2799	.78801	60
1	589	175	.2792	783	59
2	612	222	.2784	765	58
3	635	269	.2776	747	57
4	658	316	.2769	729	56
5	.61681	.78363	1.2761	.78711	55
6	704	410	.2753	694	54
7	726	457	.2746	676	53
8	749	504	.2738	658	52
9	772	551	.2731	640	51
10	.61795	.78598	1.2723	.78622	50
11	818	645	.2715	604	49
12	841	692	.2708	586	48
13	864	739	.2700	568	47
14	887	786	.2693	550	46
15	.61909	.78834	1.2685	.78532	45
16	932	881	.2677	514	44
17	955	928	.2670	496	43
18	.61978	.78975	.2662	478	42
19	.62001	.79022	.2655	460	41
20	.62024	.79070	1.2647	.78442	40
21	046	117	.2640	424	39
22	069	164	.2632	405	38
23	092	212	.2624	387	37
24	115	259	.2617	369	36
25	.62138	.79306	1.2609	.78351	35
26	160	354	.2602	333	34
27	183	401	.2594	315	33
28	206	449	.2587	297	32
29	229	496	.2579	279	31
30	.62251	.79544	1.2572	.78261	30
31	274	591	.2564	243	29
32	297	639	.2557	225	28
33	320	686	.2549	206	27
34	342	734	.2542	188	26
35	.62365	.79781	1.2534	.78170	25
36	388	829	.2527	152	24
37	411	877	.2519	134	23
38	433	924	.2512	116	22
39	456	.79972	.2504	098	21
40	.62479	.80020	1.2497	.78079	20
41	502	067	.2489	061	19
42	524	115	.2482	043	18
43	547	163	.2475	025	17
44	570	211	.2467	.78007	16
45	.62592	.80258	1.2460	.77988	15
46	615	306	.2452	970	14
47	638	354	.2445	952	13
48	660	402	.2437	934	12
49	683	450	.2430	916	11
50	.62706	.80498	1.2423	.77897	10
51	728	546	.2415	879	9
	751	594	.2408	861	8
	774	642	.2401	843	7
	796	690	.2393	824	6
	.62819	.80738	1.2386	.77806	5
	842	786	.2378	788	4
	864	834	.2371	769	3
	887	882	.2364	751	2
	909	930	.2356	733	1
	.62932	.80978	1.2349	.77715	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.62932	.80978	1.2349	.77715	60
1	955	.81027	.2342	696	59
2	.62977	075	.2334	678	58
3	.63000	123	.2327	660	57
4	022	171	.2320	641	56
5	.63045	.81220	1.2312	.77623	55
6	068	268	.2305	605	54
7	090	316	.2298	586	53
8	113	364	.2290	568	52
9	135	413	.2283	550	51
10	.63158	.81461	1.2276	.77531	50
11	180	510	.2268	513	49
12	203	558	.2261	494	48
13	225	606	.2254	476	47
14	248	655	.2247	458	46
15	.63271	.81703	1.2239	.77439	45
16	293	752	.2232	421	44
17	316	800	.2225	402	43
18	338	849	.2218	384	42
19	361	898	.2210	366	41
20	.63383	.81946	1.2203	.77347	40
21	406	.81995	.2196	329	39
22	428	.82044	.2189	310	38
23	451	092	.2181	292	37
24	473	141	.2174	273	36
25	.63496	.82190	1.2167	.77255	35
26	518	238	.2160	236	34
27	540	287	.2153	218	33
28	563	336	.2145	199	32
29	585	385	.2138	181	31
30	.63608	.82434	1.2131	.77162	30
31	630	483	.2124	144	29
32	653	531	.2117	125	28
33	675	580	.2109	107	27
34	698	629	.2102	088	26
35	.63720	.82678	1.2095	.77070	25
36	742	727	.2088	051	24
37	765	776	.2081	033	23
38	787	825	.2074	.77014	22
39	810	874	.2066	.76996	21
40	.63832	.82923	1.2059	.76977	20
41	854	.82972	.2052	959	19
42	877	.83022	.2045	940	18
43	899	071	.2038	921	17
44	922	120	.2031	903	16
45	.63944	.83169	1.2024	.76884	15
46	966	218	.2017	866	14
47	.63989	268	.2009	847	13
48	.64011	317	.2002	828	12
49	033	366	.1995	810	11
50	.64056	.83415	1.1988	.76791	10
51	078	465	.1981	772	9
52	100	514	.1974	754	8
53	123	564	.1967	735	7
54	145	613	.1960	717	6
55	.64167	.83662	1.1953	.76698	5
56	190	712	.1946	679	4
57	212	761	.1939	661	3
58	234	811	.1932	642	2
59	256	860	.1925	623	1
60	.64279	.83910	1.1918	.76604	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.64279	.83910	1.1918	.76604	60
1	301	.83960	.1910	586	59
2	323	.84009	.1903	567	58
3	346	059	.1896	548	57
4	368	108	.1889	530	56
5	.64390	.84158	1.1882	.76511	55
6	412	208	.1875	492	54
7	435	258	.1868	473	53
8	457	307	.1861	455	52
9	479	357	.1854	436	51
10	.64501	.84407	1.1847	.76417	50
11	524	457	.1840	398	49
12	546	507	.1833	380	48
13	568	556	.1826	361	47
14	590	606	.1819	342	46
15	.64612	.84656	1.1812	.76323	45
16	635	706	.1806	304	44
17	657	756	.1799	286	43
18	679	806	.1792	267	42
19	701	856	.1785	248	41
20	.64723	.84906	1.1778	.76229	40
21	746	.84956	.1771	210	39
22	768	.85006	.1764	192	38
23	790	057	.1757	173	37
24	812	107	.1750	154	36
25	.64834	.85157	1.1743	.76135	35
26	856	207	.1736	116	34
27	878	257	.1729	097	33
28	901	308	.1722	078	32
29	923	358	.1715	059	31
30	.64945	.85408	1.1708	.76041	30
31	967	458	.1702	022	29
32	.64989	509	.1695	.76003	28
33	.65011	559	.1688	.75984	27
34	033	609	.1681	965	26
35	.65055	.85660	1.1674	.75946	25
36	077	710	.1667	927	24
37	100	761	.1660	908	23
38	122	811	.1653	889	22
39	144	862	.1647	870	21
40	.65166	.85912	1.1640	.75851	20
41	188	.85963	.1633	832	19
42	210	.86014	.1626	813	18
43	232	064	.1619	794	17
44	254	115	.1612	775	16
45	.65276	.86166	1.1606	.75756	15
46	298	216	.1599	738	14
47	320	267	.1592	719	13
48	342	318	.1585	700	12
49	364	368	.1578	680	11
50	.65386	.86419	1.1571	.75661	10
51	408	470	.1565	642	9
52	430	521	.1558	623	8
53	452	572	.1551	604	7
54	474	623	.1544	585	6
55	.65496	.86674	1.1538	.75566	5
56	518	725	.1531	547	4
57	540	776	.1524	528	3
58	562	827	.1517	509	2
59	584	878	.1510	490	1
60	.65606	.86929	1.1504	.75471	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.65606	.86929	1.1504	.75471	60
1	628	.86980	.1497	452	59
2	650	.87031	.1490	433	58
3	672	082	.1483	414	57
4	694	133	.1477	395	56
5	.65716	.87184	1.1470	.75375	55
6	738	236	.1463	356	54
7	759	287	.1456	337	53
8	781	338	.1450	318	52
9	803	389	.1443	299	51
10	.65825	.87441	1.1436	.75280	50
11	847	492	.1430	261	49
12	869	543	.1423	241	48
13	891	595	.1416	222	47
14	913	646	.1410	203	46
15	.65935	.87698	1.1403	.75184	45
16	956	749	.1396	165	44
17	.65978	801	.1389	146	43
18	.66000	852	.1383	126	42
19	022	904	.1376	107	41
20	.66044	.87955	1.1369	.75088	40
21	066	.88007	.1363	069	39
22	088	059	.1356	050	38
23	109	110	.1349	030	37
24	131	162	.1343	.75011	36
25	.66153	.88214	1.1336	.74992	35
26	175	265	.1329	973	34
27	197	317	.1323	953	33
28	218	369	.1316	934	32
29	240	421	.1310	915	31
30	.66262	.88473	1.1303	.74896	30
31	284	524	.1296	876	29
32	306	576	.1290	857	28
33	327	628	.1283	838	27
34	349	680	.1276	818	26
35	.66371	.88732	1.1270	.74799	25
36	393	784	.1263	780	24
37	414	836	.1257	760	23
38	436	888	.1250	741	22
39	458	940	.1243	722	21
40	.66480	.88992	1.1237	.74703	20
41	501	.89045	.1230	683	19
42	523	097	.1224	664	18
43	545	149	.1217	644	17
44	566	201	.1211	625	16
45	.66588	.89253	1.1204	.74606	15
46	610	306	.1197	586	14
47	632	358	.1191	567	13
48	653	410	.1184	548	12
49	675	463	.1178	528	11
50	.66697	.89515	1.1171	.74509	10
51	718	567	.1165	489	9
52	740	620	.1158	470	8
53	762	672	.1152	451	7
54	783	725	.1145	431	6
55	.66805	.89777	1.1139	.74412	5
56	827	830	.1132	392	4
57	848	883	.1126	373	3
58	870	935	.1119	353	2
59	891	.89988	.1113	334	1
60	.66913	.90040	1.1106	.74314	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.66913	.90040	1.1106	.74314	60
1	935	093	.1100	295	59
2	956	146	.1093	276	58
3	978	199	.1087	256	57
4	.66999	251	.1080	237	56
5	.67021	.90304	1.1074	.74217	55
6	043	357	.1067	198	54
7	064	410	.1061	178	53
8	086	463	.1054	159	52
9	107	516	.1048	139	51
10	.67129	.90569	1.1041	.74120	50
11	151	621	.1035	100	49
12	172	674	.1028	080	48
13	194	727	.1022	061	47
14	215	781	.1016	041	46
15	.67237	.90834	1.1009	.74022	45
16	258	887	.1003	.74002	44
17	280	940	.0996	.73983	43
18	301	.90993	.0990	903	42
19	323	.91046	.0983	944	41
20	.67344	.91099	1.0977	.73924	40
21	366	153	.0971	904	39
22	387	206	.0964	885	38
23	409	259	.0958	865	37
24	430	313	.0951	846	36
25	.67452	.91366	1.0945	.73826	35
26	473	419	.0939	806	34
27	495	473	.0932	787	33
28	516	526	.0926	767	32
29	538	580	.0919	747	31
30	.67559	.91633	1.0913	.73728	30
31	580	687	.0907	708	29
32	602	740	.0900	688	28
33	623	794	.0894	669	27
34	645	847	.0888	649	26
35	.67666	.91901	1.0881	.73629	25
36	688	.91953	.0875	610	24
37	709	.92008	.0869	590	23
38	730	062	.0862	570	22
39	752	116	.0856	551	21
40	.67773	.92170	1.0850	.73531	20
41	795	224	.0843	511	19
42	816	277	.0837	491	18
43	837	331	.0831	472	17
44	859	385	.0824	452	16
45	.67880	.92439	1.0818	.73432	15
46	901	493	.0812	413	14
47	923	547	.0805	393	13
48	944	601	.0799	373	12
49	965	655	.0793	353	11
50	.67987	.92709	1.0786	.73333	10
51	.68008	763	.0780	314	9
	020	817	.0774	294	8
		872	.0768	274	7
		926	.0761	254	6
		.92980	1.0755	.73234	5
	115	.93034	.0749	215	4
	136	088	.0742	195	3
	157	143	.0736	175	2
	179	197	.0730	155	1
	.68200	.93252	1.0724	.73135	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.68200	.93252	1.0724	.73135	60
1	221	306	.0717	116	59
2	242	360	.0711	096	58
3	264	415	.0705	076	57
4	285	469	.0699	056	56
5	.68306	.93524	1.0692	.73036	55
6	327	578	.0686	.73016	54
7	349	633	.0680	.72996	53
8	370	688	.0674	976	52
9	391	742	.0668	957	51
10	.68412	.93797	1.0661	.72937	50
11	434	852	.0655	917	49
12	455	906	.0649	897	48
13	476	.93961	.0643	877	47
14	497	.94016	.0637	857	46
15	.68518	.94071	1.0630	.72837	45
16	539	125	.0624	817	44
17	561	180	.0618	797	43
18	582	235	.0612	777	42
19	603	290	.0606	757	41
20	.68624	.94345	1.0599	.72737	40
21	645	400	.0593	717	39
22	666	455	.0587	697	38
23	688	510	.0581	677	37
24	709	565	.0575	657	36
25	.68730	.94620	1.0569	.72637	35
26	751	676	.0562	617	34
27	772	731	.0556	597	33
28	793	786	.0550	577	32
29	814	841	.0544	557	31
30	.68835	.94896	1.0538	.72537	30
31	857	.94952	.0532	517	29
32	878	.95007	.0526	497	28
33	899	062	.0519	477	27
34	920	118	.0513	457	26
35	.68941	.95173	1.0507	.72437	25
36	962	229	.0501	417	24
37	.68983	284	.0495	397	23
38	.69004	340	.0489	377	22
39	025	395	.0483	357	21
40	.69046	.95451	1.0477	.72337	20
41	067	506	.0470	317	19
42	088	562	.0464	297	18
43	109	618	.0458	277	17
44	130	673	.0452	257	16
45	.69151	.95729	1.0446	.72236	15
46	172	785	.0440	216	14
47	193	841	.0434	196	13
48	214	897	.0428	176	12
49	235	.95952	.0422	156	11
50	.69256	.96008	1.0416	.72136	10
51	277	064	.0410	116	9
52	298	120	.0404	095	8
53	319	176	.0398	075	7
54	340	232	.0392	055	6
55	.69361	.96288	1.0385	.72035	5
56	382	344	.0379	.72015	4
57	403	400	.0373	.71995	3
58	424	457	.0367	974	2
59	445	513	.0361	954	1
60	.69466	.96569	1.0355	.71934	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

	N. Sin.	N. Tan.	N. Cot.	N. Cos.	
0	.69466	.96569	1.0355	.71934	60
1	487	625	.0349	914	59
2	508	681	.0343	894	58
3	529	738	.0337	873	57
4	549	794	.0331	853	56
5	.69570	.96850	1.0325	.71833	55
6	591	907	.0319	813	54
7	612	.96963	.0313	792	53
8	633	.97020	.0307	772	52
9	654	076	.0301	752	51
10	.69675	.97133	1.0295	.71732	60
11	696	189	.0289	711	49
12	717	246	.0283	691	48
13	737	302	.0277	671	47
14	758	359	.0271	650	46
15	.69779	.97416	1.0265	.71630	45
16	800	472	.0259	610	44
17	821	529	.0253	590	43
18	842	586	.0247	569	42
19	862	643	.0241	549	41
20	.69883	.97700	1.0235	.71529	40
21	904	756	.0230	508	39
22	925	813	.0224	488	38
23	946	870	.0218	468	37
24	966	927	.0212	447	36
25	.69987	.97984	1.0206	.71427	35
26	.70008	.98041	.0200	407	34
27	029	098	.0194	386	33
28	049	155	.0188	366	32
29	070	213	.0182	345	31
30	.70091	.98270	1.0176	.71325	30
31	112	327	.0170	305	29
32	132	384	.0164	284	28
33	153	441	.0158	264	27
34	174	499	.0152	243	26
35	.70195	.98456	1.0147	.71223	25
36	215	613	.0141	203	24
37	236	671	.0135	182	23
38	257	728	.0129	162	22
39	277	786	.0123	141	21
40	.70298	.98843	1.0117	.71121	20
41	319	901	.0111	100	19
42	339	.98958	.0105	080	18
43	360	.99016	.0099	059	17
44	381	073	.0094	039	16
45	.70401	.99131	1.0088	.71019	15
46	422	189	.0082	.70998	14
47	443	247	.0076	978	13
48	463	304	.0070	957	12
49	484	362	.0064	937	11
50	.70505	.99420	1.0058	.70916	10
51	525	478	.0052	896	9
52	546	536	.0047	875	8
53	567	594	.0041	853	7
54	587	652	.0035	834	6
55	.70608	.99710	1.0029	.70813	5
56	628	768	.0023	793	4
57	649	826	.0017	772	3
58	670	884	.0012	752	2
59	690	.99942	.0006	731	1
60	.70711	1.0000	1.0000	.70711	0
	N. Cos.	N. Cot.	N. Tan.	N. Sin.	

DEGREES.				MINUTES.		SECONDS.			
0°	0.00000 00	60°	1.04719 76	120°	2.09439 51	0'	0.00000 00	0''	0.00000 00
1	0.01745 33	61	1.06465 08	121	2.11184 84	1	0.00029 09	1	0.00000 48
2	0.03490 66	62	1.08210 41	122	2.12930 17	2	0.00058 18	2	0.00000 97
3	0.05235 99	63	1.09955 74	123	2.14675 50	3	0.00087 27	3	0.00001 45
4	0.06981 32	64	1.11701 07	124	2.16420 83	4	0.00116 36	4	0.00001 94
5	0.08726 65	65	1.13446 40	125	2.18166 16	5	0.00145 44	5	0.00002 42
6	0.10471 98	66	1.15191 73	126	2.19911 49	6	0.00174 53	6	0.00002 91
7	0.12217 30	67	1.16937 06	127	2.21656 82	7	0.00203 62	7	0.00003 39
8	0.13962 63	68	1.18682 39	128	2.23402 14	8	0.00232 71	8	0.00003 88
9	0.15707 96	69	1.20427 72	129	2.25147 47	9	0.00261 80	9	0.00004 36
10	0.17453 29	70	1.22173 05	130	2.26892 80	10	0.00290 89	10	0.00004 85
11	0.19198 62	71	1.23918 38	131	2.28638 13	11	0.00319 98	11	0.00005 33
12	0.20943 95	72	1.25663 71	132	2.30383 46	12	0.00349 07	12	0.00005 82
13	0.22689 28	73	1.27409 04	133	2.32128 79	13	0.00378 15	13	0.00006 30
14	0.24434 61	74	1.29154 36	134	2.33874 12	14	0.00407 24	14	0.00006 79
15	0.26179 94	75	1.30899 69	135	2.35619 45	15	0.00436 33	15	0.00007 27
16	0.27925 27	76	1.32645 02	136	2.37364 78	16	0.00465 42	16	0.00007 76
17	0.29670 60	77	1.34390 35	137	2.39110 11	17	0.00494 51	17	0.00008 24
18	0.31415 93	78	1.36135 68	138	2.40855 44	18	0.00523 60	18	0.00008 73
19	0.33161 26	79	1.37881 01	139	2.42600 77	19	0.00552 69	19	0.00009 21
20	0.34906 59	80	1.39626 34	140	2.44346 10	20	0.00581 78	20	0.00009 70
21	0.36651 91	81	1.41371 67	141	2.46091 42	21	0.00610 87	21	0.00010 18
22	0.38397 24	82	1.43117 00	142	2.47836 75	22	0.00639 95	22	0.00010 67
23	0.40142 57	83	1.44862 33	143	2.49582 08	23	0.00669 04	23	0.00011 15
24	0.41887 90	84	1.46607 66	144	2.51327 41	24	0.00698 13	24	0.00011 64
25	0.43633 23	85	1.48352 99	145	2.53072 74	25	0.00727 22	25	0.00012 12
26	0.45378 56	86	1.50098 32	146	2.54818 07	26	0.00756 31	26	0.00012 61
27	0.47123 89	87	1.51843 64	147	2.56563 40	27	0.00785 40	27	0.00013 09
28	0.48869 22	88	1.53588 97	148	2.58308 73	28	0.00814 49	28	0.00013 57
29	0.50614 55	89	1.55334 30	149	2.60054 06	29	0.00843 58	29	0.00014 06
30	0.52359 88	90	1.57079 63	150	2.61799 39	30	0.00872 66	30	0.00014 54
31	0.54105 21	91	1.58824 96	151	2.63544 72	31	0.00901 75	31	0.00015 03
32	0.55850 54	92	1.60570 29	152	2.65290 05	32	0.00930 84	32	0.00015 51
33	0.57595 87	93	1.62315 62	153	2.67035 38	33	0.00959 93	33	0.00016 00
34	0.59341 19	94	1.64060 95	154	2.68780 70	34	0.00989 02	34	0.00016 48
35	0.61086 52	95	1.65806 28	155	2.70526 03	35	0.01018 11	35	0.00016 97
36	0.62831 85	96	1.67551 61	156	2.72271 36	36	0.01047 20	36	0.00017 45
37	0.64577 18	97	1.69296 94	157	2.74016 69	37	0.01076 29	37	0.00017 94
38	0.66322 51	98	1.71042 27	158	2.75762 02	38	0.01105 38	38	0.00018 42
39	0.68067 84	99	1.72787 60	159	2.77507 35	39	0.01134 46	39	0.00018 91
40	0.69813 17	100	1.74532 93	160	2.79252 68	40	0.01163 55	40	0.00019 39
41	0.71558 50	101	1.76278 25	161	2.80998 01	41	0.01192 64	41	0.00019 88
42	0.73303 83	102	1.78023 58	162	2.82743 34	42	0.01221 73	42	0.00020 36
43	0.75049 16	103	1.79768 91	163	2.84488 67	43	0.01250 82	43	0.00020 85
44	0.76794 49	104	1.81514 24	164	2.86234 00	44	0.01279 91	44	0.00021 33
45	0.78539 82	105	1.83259 57	165	2.87979 33	45	0.01309 00	45	0.00021 82
46	0.80285 15	106	1.85004 90	166	2.89724 66	46	0.01338 09	46	0.00022 30
47	0.82030 47	107	1.86750 23	167	2.91469 99	47	0.01367 17	47	0.00022 79
48	0.83775 80	108	1.88495 56	168	2.93215 31	48	0.01396 26	48	0.00023 27
49	0.85521 13	109	1.90240 89	169	2.94960 64	49	0.01425 35	49	0.00023 76
50	0.87266 46	110	1.91986 22	170	2.96705 97	50	0.01454 44	50	0.00024 24
51	0.89011 79	111	1.93731 55	171	2.98451 30	51	0.01483 53	51	0.00024 73
52	0.90757 12	112	1.95476 88	172	3.00196 63	52	0.01512 62	52	0.00025 21
53	0.92502 45	113	1.97222 21	173	3.01941 96	53	0.01541 71	53	0.00025 70
54	0.94247 78	114	1.98967 53	174	3.03687 29	54	0.01570 80	54	0.00026 18
55	0.95993 11	115	2.00712 86	175	3.05432 62	55	0.01599 89	55	0.00026 66
56	0.97738 44	116	2.02458 19	176	3.07177 95	56	0.01628 97	56	0.00027 15
57	0.99483 77	117	2.04203 52	177	3.08923 28	57	0.01658 06	57	0.00027 63
58	1.01229 10	118	2.05948 85	178	3.10668 61	58	0.01687 15	58	0.00028 12
59	1.02974 43	119	2.07694 18	179	3.12413 94	59	0.01716 24	59	0.00028 60
60	1.04719 76	120	2.09439 51	180	3.14159 27	60	0.01745 33	60	0.00029 09
DEGREES.				MINUTES.		SECONDS.			

Base of common logarithms	= 10.
Base of Napierian logarithms (e)	= 2.71828 18284 59045 23536
Com. Log. $e = M$ (Modulus of Com. Logs.)	= 0.43429 44819 03251 82765
Nap. Log. $10 = \frac{1}{M}$	= 2.30258 50929 94045 68402
Com. Log. $N = M \times \text{Nap. Log. } N$	} where N denotes any number.
Nap. Log. $N = \frac{1}{M} \times \text{Com. Log. } N$	

Multiples of M.				Multiples of $\frac{1}{M}$			
0	0.00000 000	50	21.71472 410	0	0.00000 000	50	115.12925 465
1	0.43429 448	51	22.14901 858	1	2.30258 509	51	117.43183 974
2	0.86858 896	52	22.58331 306	2	4.60517 019	52	119.73442 484
3	1.30288 345	53	23.01760 754	3	6.90775 528	53	122.03700 993
4	1.73717 793	54	23.45190 202	4	9.21034 037	54	124.33959 502
5	2.17147 241	55	23.88619 650	5	11.51292 546	55	126.64218 011
6	2.60576 689	56	24.32049 099	6	13.81551 056	56	128.94476 521
7	3.04006 137	57	24.75478 547	7	16.11809 565	57	131.24735 030
8	3.47435 586	58	25.18907 995	8	18.42068 074	58	133.54993 539
9	3.90865 034	59	25.62337 443	9	20.72326 584	59	135.85252 049
10	4.34294 482	60	26.05766 891	10	23.02585 093	60	138.15510 558
11	4.77723 930	61	26.49196 340	11	25.32843 602	61	140.45769 067
12	5.21153 378	62	26.92625 788	12	27.63102 112	62	142.76027 577
13	5.64582 826	63	27.36055 236	13	29.93360 621	63	145.06286 086
14	6.08012 275	64	27.79484 684	14	32.23619 130	64	147.36544 595
15	6.51441 723	65	28.22914 132	15	34.53877 639	65	149.66803 104
16	6.94871 171	66	28.66343 581	16	36.84136 149	66	151.97061 614
17	7.38300 619	67	29.09773 029	17	39.14394 658	67	154.27320 123
18	7.81730 067	68	29.53202 477	18	41.44653 167	68	156.57578 632
19	8.25159 516	69	29.96631 925	19	43.74911 677	69	158.87837 142
20	8.68588 964	70	30.40061 373	20	46.05170 186	70	161.18095 651
21	9.12018 412	71	30.83490 822	21	48.35428 695	71	163.48354 160
22	9.55447 860	72	31.26920 270	22	50.65687 203	72	165.78612 670
23	9.98877 308	73	31.70349 718	23	52.95945 714	73	168.08871 179
24	10.42306 757	74	32.13779 166	24	55.26204 223	74	170.39129 688
25	10.85736 205	75	32.57208 614	25	57.56462 732	75	172.69388 197
26	11.29165 653	76	33.00638 062	26	59.86721 242	76	174.99646 707
27	11.72595 101	77	33.44067 511	27	62.16979 751	77	177.29905 216
28	12.16024 549	78	33.87496 959	28	64.47238 260	78	179.60163 725
29	12.59453 998	79	34.30926 407	29	66.77496 770	79	181.90422 235
30	13.02883 446	80	34.74355 855	30	69.07755 279	80	184.20680 744
31	13.46312 894	81	35.17785 303	31	71.38013 788	81	186.50939 253
32	13.89742 342	82	35.61214 752	32	73.68272 298	82	188.81197 763
33	14.33171 790	83	36.04644 200	33	75.98530 807	83	191.11456 272
34	14.76601 238	84	36.48073 648	34	78.28789 316	84	193.41714 781
35	15.20030 687	85	36.91503 096	35	80.59047 825	85	195.71973 290
36	15.63460 135	86	37.34932 544	36	82.89306 335	86	198.02231 800
37	16.06889 583	87	37.78361 993	37	85.19564 844	87	200.32490 309
38	16.50319 031	88	38.21791 441	38	87.49823 353	88	202.62748 818
39	16.93748 479	89	38.65220 889	39	89.80081 863	89	204.93007 328
40	17.37177 928	90	39.08650 337	40	92.10340 372	90	207.23265 827
41	17.80607 376	91	39.52079 785	41	94.40598 881	91	209.53524 3
42	18.24036 824	92	39.95509 234	42	96.70857 391	92	211.83782 8
43	18.67466 272	93	40.38938 682	43	99.01115 900	93	214.14041 3
44	19.10895 720	94	40.82368 130	44	101.31374 409	94	216.44299 8
45	19.54325 169	95	41.25797 578	45	103.61632 918	95	218.74558 3
46	19.97754 617	96	41.69227 026	46	105.91891 428	96	221.04816 8
47	20.41184 065	97	42.12656 474	47	108.22149 937	97	223.35075 4
48	20.84613 513	98	42.56085 923	48	110.52408 446	98	225.65333 9
49	21.28042 961	99	42.99515 371	49	112.82666 956	99	227.95592 4
50	21.71472 410	100	43.42944 819	50	115.12925 465	100	230.25850 9

TRIGONOMETRIC FORMULAS.

$$\sin^2 \alpha + \cos^2 \alpha = 1.$$

$$\sec^2 \alpha = 1 + \tan^2 \alpha.$$

$$\operatorname{cosec}^2 \alpha = 1 + \cot^2 \alpha.$$

$$\tan \alpha = \frac{\sin \alpha}{\cos \alpha}.$$

$$\cot \alpha = \frac{\cos \alpha}{\sin \alpha}.$$

$$\sec \alpha = \frac{1}{\cos \alpha}.$$

$$\operatorname{cosec} \alpha = \frac{1}{\sin \alpha}.$$

$$\sin \alpha = \pm \frac{\tan \alpha}{\sqrt{1 + \tan^2 \alpha}}.$$

$$\cos \alpha = \pm \frac{1}{\sqrt{1 + \tan^2 \alpha}}.$$

$$\sin (\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta.$$

$$\cos (\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta.$$

$$\tan (\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}.$$

$$\sin \alpha + \sin \beta = 2 \sin \frac{1}{2} (\alpha + \beta) \cos \frac{1}{2} (\alpha - \beta).$$

$$\sin \alpha - \sin \beta = 2 \cos \frac{1}{2} (\alpha + \beta) \sin \frac{1}{2} (\alpha - \beta).$$

$$\cos \alpha + \cos \beta = 2 \cos \frac{1}{2} (\alpha + \beta) \cos \frac{1}{2} (\alpha - \beta).$$

$$\cos \alpha - \cos \beta = -2 \sin \frac{1}{2} (\alpha + \beta) \sin \frac{1}{2} (\alpha - \beta).$$

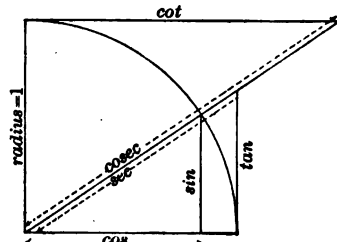


FIG. 1.

$$\sin \alpha \sin \beta = \frac{1}{2} \cos (\alpha - \beta) - \frac{1}{2} \cos (\alpha + \beta).$$

$$\cos \alpha \cos \beta = \frac{1}{2} \cos (\alpha - \beta) + \frac{1}{2} \cos (\alpha + \beta).$$

$$\sin \alpha \cos \beta = \frac{1}{2} \sin (\alpha + \beta) + \frac{1}{2} \sin (\alpha - \beta).$$

$$\sin^2 \alpha - \sin^2 \beta = \cos^2 \beta - \cos^2 \alpha = \sin (\alpha + \beta) \sin (\alpha - \beta).$$

$$\cos^2 \alpha - \sin^2 \beta = \cos^2 \beta - \sin^2 \alpha = \cos (\alpha + \beta) \cos (\alpha - \beta).$$

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha.$$

$$\cos 2\alpha = \cos^2 \alpha - \sin^2 \alpha.$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}.$$

$$2 \sin^2 \frac{1}{2} \alpha = 1 - \cos \alpha.$$

$$2 \cos^2 \frac{1}{2} \alpha = 1 + \cos \alpha.$$

$$\tan \frac{1}{2} \alpha = \pm \sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}} = \frac{\sin \alpha}{1 + \cos \alpha} = \frac{1 - \cos \alpha}{\sin \alpha}.$$

$$\sin \alpha + \sin (\alpha + x) + \sin (\alpha + 2x) + \dots + \sin (\alpha + nx)$$

$$= \frac{\sin \frac{1}{2} (n+1)x \sin (\alpha + \frac{1}{2} nx)}{\sin \frac{1}{2} x}.$$

$$\cos \alpha + \cos (\alpha + x) + \cos (\alpha + 2x) + \dots + \cos (\alpha + nx)$$

$$= \frac{\sin \frac{1}{2} (n+1)x \cos (\alpha + \frac{1}{2} nx)}{\sin \frac{1}{2} x}.$$

$$-1.$$

$$e^{xi} = \cos x + i \sin x.$$

$$e^{-xi} = \cos x - i \sin x.$$

$$\cos x = \frac{1}{2} (e^{xi} + e^{-xi}).$$

$$\sin x = \frac{1}{2i} (e^{xi} - e^{-xi}).$$

$$e^{nxi} = (\cos x + i \sin x)^n = \cos nx + i \sin nx.$$

PLANE TRIANGLES.

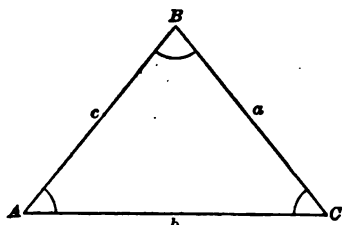


FIG. 2.

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$a = b \cos C + c \cos B.$$

$$a^2 = b^2 + c^2 - 2bc \cos A.$$

$$a \sin \frac{1}{2}(B - C) = (b - c) \cos \frac{1}{2}A.$$

$$a \cos \frac{1}{2}(B - C) = (b + c) \sin \frac{1}{2}A.$$

$$\frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}, \quad \tan A = \frac{a \sin B}{c - a \cos B}.$$

$$c = \frac{a-b}{\cos x}, \text{ if } \tan x = \frac{2 \sin \frac{1}{2}C}{a-b} \sqrt{ab}.$$

$$\text{If } s = \frac{1}{2}(a+b+c):$$

$$\sin \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}}, \quad \cos \frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}}, \quad \tan \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}, \quad \tan \frac{1}{2}A = \frac{r}{s-a}, \quad \tan \frac{1}{2}B = \frac{r}{s-b}, \quad \tan \frac{1}{2}C = \frac{r}{s-c}.$$

$$\text{Area} = \frac{1}{2}ab \sin C = \frac{c^2}{2} \cdot \frac{\sin A \sin B}{\sin C} = \sqrt{s(s-a)(s-b)(s-c)}.$$

Radius of inscribed circle = r .

$$\text{Diameter of circumscribed circle} = \frac{a}{\sin A}.$$

DIFFERENTIAL FORMULAS FOR PLANE TRIANGLES.

$$dA + dB + dC = 0.$$

$$\frac{da}{a} - \cot A dA = \frac{db}{b} - \cot B dB = \frac{dc}{c} - \cot C dC.$$

$$da = \cos C db + \cos B dc + b \sin C dA.$$

$$a dB = \sin C db - \sin B dc - b \cos C dA.$$

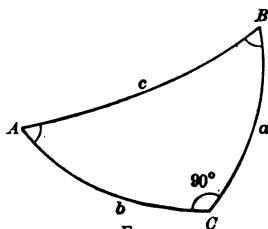
RIGHT SPHERICAL TRIANGLES ($C = 90^\circ$).

FIG. 3.

$$\sin a = \sin A \sin c.$$

$$\sin a = \cot B \tan b.$$

$$\cos A = \sin B \cos a.$$

$$\cos A = \tan b \cot c.$$

$$\sin b = \sin B \sin c.$$

$$\sin b = \cot A \tan c.$$

$$\cos B = \sin A$$

$$\cos B = \tan a$$

$$\cos c = \cos a \cos b = \cot A \cot B.$$

OBLIQUE SPHERICAL TRIANGLES.

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}.$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A.$$

$$\sin A \cos b = \cos B \sin C + \sin B \cos C \cos a.$$

$$\sin a \cos b = \sin c \cos B + \cos a \sin b \cos C.$$

$$\sin A \cos B = \cos b \sin C - \cos c \cos A \sin B.$$

$$\sin a \cot b = \cot B \sin C + \cos a \cot C.$$

$$\sin A \cot B = \cot b \sin c - \cos c \cos A.$$

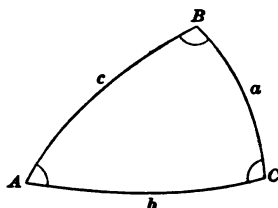


FIG. 4.

$$s = \frac{1}{2}(a + b + c).$$

$$\sin^2 \frac{1}{2} A = \frac{\sin(s-b) \sin(s-c)}{\sin b \sin c}.$$

$$\cos^2 \frac{1}{2} A = \frac{\sin s \sin(s-a)}{\sin b \sin c}.$$

$$\tan^2 \frac{1}{2} A = \frac{\sin(s-b) \sin(s-c)}{\sin s \sin(s-a)}.$$

$$r^2 = \frac{\sin(s-a) \sin(s-b) \sin(s-c)}{\sin s}.$$

$$\tan \frac{1}{2} A = \frac{r}{\sin(s-a)}.$$

$$S = \frac{1}{2}(A + B + C).$$

$$\sin^2 \frac{1}{2} a = \frac{-\cos S \cos(S-A)}{\sin B \sin C}.$$

$$\cos^2 \frac{1}{2} a = \frac{\cos(S-B) \cos(S-C)}{\sin B \sin C}.$$

$$\tan^2 \frac{1}{2} a = \frac{-\cos S \cos(S-A)}{\cos(S-B) \cos(S-C)}.$$

$$R^2 = \frac{-\cos S}{\cos(S-A) \cos(S-B) \cos(S-C)}.$$

$$\tan \frac{1}{2} a = R \cos(S-A).$$

$$\sin \frac{1}{2} c \sin \frac{1}{2} (A-B) = \cos \frac{1}{2} C \sin \frac{1}{2} (a-b).$$

$$\sin \frac{1}{2} c \cos \frac{1}{2} (A-B) = \sin \frac{1}{2} C \sin \frac{1}{2} (a+b).$$

$$\cos \frac{1}{2} c \sin \frac{1}{2} (A+B) = \cos \frac{1}{2} C \cos \frac{1}{2} (a-b).$$

$$\cos \frac{1}{2} c \cos \frac{1}{2} (A+B) = \sin \frac{1}{2} C \cos \frac{1}{2} (a+b).$$

$$\frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a-b)} = \frac{\sin \frac{1}{2} (A+B)}{\sin \frac{1}{2} (A-B)}.$$

$$\frac{\cot \frac{1}{2} C}{\tan \frac{1}{2} (A-B)} = \frac{\sin \frac{1}{2} (a+b)}{\sin \frac{1}{2} (a-b)}.$$

$$\frac{\tan \frac{1}{2} c}{\tan \frac{1}{2} (a+b)} = \frac{\cos \frac{1}{2} (A+B)}{\cos \frac{1}{2} (A-B)}.$$

$$\frac{\cot \frac{1}{2} C}{\tan \frac{1}{2} (A+B)} = \frac{\cos \frac{1}{2} (a+b)}{\cos \frac{1}{2} (a-b)}.$$

r = tangent of the angular radius of the inscribed small circle.

R = tangent of the angular radius of the circumscribed small circle.

SPHERICAL EXCESS.

$$E = A + B + C - 180^\circ.$$

$$\sin \frac{1}{2} E = \frac{\sin \frac{1}{2} a \sin \frac{1}{2} b}{\cos \frac{1}{2} c} \sin C.$$

$$\tan \frac{1}{2} E = \frac{\tan \frac{1}{2} a \tan \frac{1}{2} b \sin C}{1 + \tan \frac{1}{2} a \tan \frac{1}{2} b \cos C}.$$

$$\tan^2 \frac{1}{2} E = \tan \frac{1}{2} s \tan \frac{1}{2} (s-a) \tan \frac{1}{2} (s-b) \tan \frac{1}{2} (s-c).$$

$$E'' = \text{area} + r^2 \sin I''.$$

DIFFERENTIAL FORMULAS FOR SPHERICAL TRIANGLES

$$\cot a \, da - \cot A \, dA = \cot b \, db - \cot B \, dB = \cot c \, dc - \cot C \, dC.$$

$$da = \cos C \, db + \cos B \, dc + \sin c \sin B \, dA.$$

$$dA = \sin b \sin C \, da - \cos c \, dB - \cos b \, dC.$$

$$\sin c \, dB = -\cos c \sin B \, da + \sin A \, db - \sin b \cos A \, dC.$$

MACLAURIN'S THEOREM.*

$$f(x) = f(0) + f'(0) \frac{x}{1} + f''(0) \frac{x^2}{2!} + f'''(0) \frac{x^3}{3!} + \dots$$

TAYLOR'S THEOREM.*

$$f(x+h) = f(x) + f'(x) \frac{h}{1} + f''(x) \frac{h^2}{2!} + f'''(x) \frac{h^3}{3!} + \dots$$

$$f(x+h, y+k) = f(x, y) + \frac{du}{dx} \frac{h}{1} + \frac{du}{dy} \frac{k}{1} + \frac{d^2u}{dx^2} \frac{h^2}{2!} + \frac{d^2u}{dy^2} \frac{k^2}{2!} + \frac{d^2u}{dx \, dy} h k + \dots,$$

where $u = f(x, y)$.

LAGRANGE'S THEOREM.*

$$u = f(z), \text{ and } z = y + x \phi(z);$$

$$u = f(y) + \phi(y) \frac{df(y)}{dy} \frac{x}{1} + \frac{d \left[\phi(y)^2 \frac{df(y)}{dy} \right]}{dy} \frac{x^2}{2!} + \frac{d^2 \left[\phi(y)^3 \frac{df(y)}{dy} \right]}{dy^2} \frac{x^3}{3!} + \dots$$

BINOMIAL THEOREM.

$$(a \pm b)^n = a^n \pm \frac{n}{1} a^{n-1} b + \frac{n(n-1)}{1 \cdot 2} a^{n-2} b^2 \pm \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} a^{n-3} b^3 + \dots$$

EXPONENTIAL THEOREM.*

$$e^x = 1 + \frac{\log a}{M} x + \left(\frac{\log a}{M} \right)^2 \frac{x^2}{2!} + \left(\frac{\log a}{M} \right)^3 \frac{x^3}{3!} + \left(\frac{\log a}{M} \right)^4 \frac{x^4}{4!} + \dots$$

$$e^x = 1 + x + \frac{1}{2} x^2 + \frac{1}{6} x^3 + \frac{1}{24} x^4 + \frac{1}{120} x^5 + \frac{1}{720} x^6 + \dots$$

* $n!$ denotes "factorial n ," or the product $1 \cdot 2 \cdot 3 \cdot 4 \dots$

$$\log(a+b) = \log a + M \left[\frac{1}{1} \left(\frac{b}{a} \right) - \frac{1}{2} \left(\frac{b}{a} \right)^2 + \frac{1}{3} \left(\frac{b}{a} \right)^3 - \frac{1}{4} \left(\frac{b}{a} \right)^4 + \dots \right]$$

$$\log(a+b) = \log a + 2M \left[\frac{1}{1} \left(\frac{b}{2a+b} \right) + \frac{1}{3} \left(\frac{b}{2a+b} \right)^3 + \frac{1}{5} \left(\frac{b}{2a+b} \right)^5 + \dots \right]$$

$$\log(1+x) = M \left(x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{5}x^5 - \dots \right).$$

$$\log(1-x) = -M \left(x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{1}{4}x^4 + \frac{1}{5}x^5 + \dots \right).$$

$$\log \frac{a}{b} = 2M \left[\frac{1}{1} \left(\frac{a-b}{a+b} \right) + \frac{1}{3} \left(\frac{a-b}{a+b} \right)^3 + \frac{1}{5} \left(\frac{a-b}{a+b} \right)^5 + \dots \right]$$

$$a = 1 + \frac{1}{1} \left(\frac{\log a}{M} \right) + \frac{1}{2!} \left(\frac{\log a}{M} \right)^2 + \frac{1}{3!} \left(\frac{\log a}{M} \right)^3 + \frac{1}{4!} \left(\frac{\log a}{M} \right)^4 + \dots$$

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$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + \frac{62}{2835}x^9 + \frac{1382}{155925}x^{11} + \dots$$

$$\cot x = \frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \frac{1}{4725}x^7 - \frac{2}{93555}x^9 - \dots$$

$$\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \frac{277}{8064}x^8 + \dots$$

$$\operatorname{cosec} x = \frac{1}{x} + \frac{1}{6}x + \frac{7}{360}x^3 + \frac{31}{15120}x^5 + \frac{127}{604800}x^7 + \dots$$

$$\sin^{-1} y = y + \frac{1}{6}y^3 + \frac{3}{40}y^5 + \frac{5}{112}y^7 + \frac{35}{1152}y^9 + \dots$$

$$\cos^{-1} y = \frac{\pi}{2} - y - \frac{1}{6}y^3 - \frac{3}{40}y^5 - \frac{5}{112}y^7 - \frac{35}{1152}y^9 - \dots$$

$$\tan^{-1} y = y - \frac{1}{3}y^3 + \frac{1}{5}y^5 - \frac{1}{7}y^7 + \frac{1}{9}y^9 - \dots$$

$$\cot^{-1} y = \frac{1}{y} - \frac{1}{3y^3} + \frac{1}{5y^5} - \frac{1}{7y^7} + \frac{1}{9y^9} - \dots$$

$$\log \sin x = \log x - M \left(\frac{1}{6}x^2 + \frac{1}{180}x^4 + \frac{1}{2835}x^6 + \frac{1}{37800}x^8 + \dots \right).$$

$$\log \cos x = -M \left(\frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \frac{17}{2520}x^8 + \dots \right).$$

$$\log \tan x = \log x + M \left(\frac{1}{3}x^2 + \frac{7}{90}x^4 + \frac{62}{2835}x^6 + \frac{127}{18900}x^8 + \dots \right).$$

$$\log \sin^{-1} y = \log y + M \left(\frac{1}{6}y^2 + \frac{11}{180}y^4 + \frac{191}{5670}y^6 + \dots \right).$$

$$\log \tan^{-1} y = \log y - M \left(\frac{1}{3}y^2 - \frac{13}{90}y^4 + \frac{251}{2835}y^6 - \dots \right).$$

$$\tan x = M \left(\frac{1}{2}\tan^2 x - \frac{1}{4}\tan^4 x + \frac{1}{6}\tan^6 x - \frac{1}{8}\tan^8 x + \dots \right).$$

$$\sin x = M \left(\frac{1}{2}\sin^2 x + \frac{1}{4}\sin^4 x + \frac{1}{6}\sin^6 x + \frac{1}{8}\sin^8 x + \dots \right).$$

! denotes "factorial n ," or the product $1 \cdot 2 \cdot 3 \cdot 4 \dots n$.

the angles are expressed in circular measure.

DIFFERENTIATION.

$$d(ax + b) = a \, dx.$$

$$d(u \pm v) = du \pm dv.$$

$$d(uv) = u \, dv + v \, du.$$

$$d\left(\frac{x}{y}\right) = \frac{y \, dx - x \, dy}{y^2}.$$

$$d(x^n) = nx^{n-1} \, dx.$$

$$d(\sqrt{x}) = \frac{dx}{2\sqrt{x}}.$$

$$d(\log x) = \frac{dx}{x}.$$

$$d(a^x) = \frac{1}{M} a^x \log a \, dx.$$

$$d(e^x) = e^x \, dx.$$

$$d(x^y) = x^y \log_e x \, dy + yx^{y-1} \, dx.$$

$$d(\sin x) = \cos x \, dx.$$

$$d(\cos x) = -\sin x \, dx.$$

$$d(\tan x) = \sec^2 x \, dx.$$

$$d(\cot x) = -\operatorname{cosec}^2 x \, dx.$$

$$d(\sec x) = \sec x \tan x \, dx.$$

$$d(\operatorname{cosec} x) = -\operatorname{cosec} x \cot x \, dx.$$

$$d(\sin^{-1} x) = \frac{dx}{\sqrt{1-x^2}}.$$

$$d(\tan^{-1} x) = \frac{dx}{1+x^2}.$$

$$d(\sec^{-1} x) = \frac{dx}{x\sqrt{x^2-1}}.$$

$$d(\cos^{-1} x) = -\frac{dx}{\sqrt{1-x^2}}.$$

$$d(\cot^{-1} x) = -\frac{dx}{1+x^2}.$$

$$d(\operatorname{cosec}^{-1} x) = -\frac{dx}{x\sqrt{x^2-1}}.$$

$$d(\operatorname{vers}^{-1} x) = \frac{dx}{\sqrt{2x-x^2}}.$$

$$d(\operatorname{covers}^{-1} x) = -\frac{dx}{\sqrt{2x-x^2}}.$$

APPROXIMATE INTEGRATION.

Let Δx be the common distance between the ordinates $y_0, y_1, y_2, \dots, y_n$, where n is even.

$$1. \int f(x) \, dx = \frac{P+2Q}{3} - \frac{y_n''' - y_0'''}{180} \Delta x^4 + \frac{y_n^v - y_0^v}{1512} \Delta x^6 - \dots,$$

where $P = \Delta x [y_0 + y_n + 2(y_2 + y_4 + \dots + y_{n-2})],$

$$Q = 2 \Delta x [y_1 + y_3 + \dots + y_{n-1}],$$

$$Y_n''' = \frac{d^3}{dx^3} f(x) \text{ when } x = \text{abscissa of } y_n.$$

2. Simpson's rule :

$$A = \frac{\Delta x}{3} [y_0 + y_n + 2(y_2 + y_4 + \dots + y_{n-2}) + 4(y_1 + y_3 + \dots + y_{n-1})].$$

3. Weddle's rule (for seven ordinates) :

$$A = \frac{3 \Delta x}{10} [y_0 + y_2 + y_4 + y_6 + y_8 + 5(y_1 + y_3 + y_5)].$$

4. Prismoidal formula : $V = \frac{\Delta x}{3} [A + A' + 4 A_m] = \frac{h}{6} [A + A' + 4 A_m].$

CONSTANTS.

LOGARITHMS.

Base of Napierian logs: $e =$	2.71828 183	0.43429 448
Modulus of common logs: $\log e = M =$	0.43429 448	9.63778 431 - 10
Degrees in arc = radius: $180^\circ \div \pi =$	$57^\circ.29577\ 951$	1.75812 263
Minutes in arc = radius:	$3\ 437'.74677$	3.53627 388
Seconds in arc = radius:	$206\ 264''.806$	5.31442 513
360° expressed in minutes of arc:	$21\ 600'$	4.33445 375
360° expressed in seconds of arc:	$1\ 296\ 000''$	6.11260 500
24 hours expressed in minutes of time:	$1\ 440^m$	3.15836 249
24 hours expressed in seconds of time:	$86\ 400^s$	4.93651 374
$\pi = 3.14159\ 26535\ 89793\ 23846$		0.49714 987
$\log \pi = 0.49714\ 98726\ 94133\ 85435$		
$\sin 1'' = 0.00000\ 48481\ 36811\ 07637$		4.68557 487 - 10
$\arcsin 1'' = 0.00000\ 48481\ 36811\ 09536$		4.68557 487 - 10

LOGARITHMS.

1 Eng. inch	0.02540 01	meters	8.40483 5 - 10
1 Eng. foot	0.30480 1	meters	9.48401 6 - 10
1 Eng. yard	0.91440 2	meters	9.96113 7 - 10
1 Eng. statute mile	1.60935	kilometers	0.20665 0
1 meter	39.3700	Eng. inches	1.59516 5
1 meter	3.28083	Eng. feet	0.51598 4
1 meter	1.09361 1	Eng. yards	0.03886 3
1 kilometer	0.62137 0	Eng. statute miles	9.79335 0 - 10
1 sq. foot	9.29034	sq. decimeters	0.96803 2
1 sq. inch	6.45163	sq. centimeters	0.80966 9
1 sq. meter	10.7639	sq. feet	1.03196 8
1 sq. centimeter	0.15500 0	sq. inches	9.19033 1 - 10
1 cubic foot	0.02831 70	cubic meters	8.45204 7 - 10
1 cubic inch	16.3872	cubic centimeters	1.21450 4
1 cubic meter	35.3145	cubic feet	1.54795 3
1 cubic decimeter (liter)	61.0234	cubic inches	1.78549 6
1 avoirdupois pound	453.59242 77	grams	2.65666 6
1 avoirdupois ounce	28.34953	grams	1.45254 6
1 Troy ounce	31.10348	grams	1.49280 9
1 grain	64.79892	milligrams	1.81156 8
1 kilogram	2.20462	avdp. pounds	0.34333 4
1 kilogram	35.2740	avdp. ounces	1.54745 4
1 kilogram	32.1507	Troy ounces	1.50719 1
1 gram	15.43235 639	grains	1.18843 2
1 foot-pound	0.13825 5	kilogram-meters	9.14068 2 - 10
1 kilogram-meter	7.23300	foot-pounds	0.85931 8
1 pound per sq. in.	70.3067	grams per sq. cm.	1.84699 7
1 gram per sq. cm.	0.01422 34	lbs. per sq. in.	8.15300 3 - 10
1 pound per cu. ft.	0.01601 84	grams per cu. cm.	8.20461 8 - 10
1 grain per cu. in.	0.00395 425	grams per cu. cm.	7.59706 4 - 10
1 gram per cu. cm.	62.4283	lbs. per cu. ft.	1.79538 2
1 gram per cu. cm.	252.8925	grains per cu. in.	2.40293 6

LOGARITHMS.

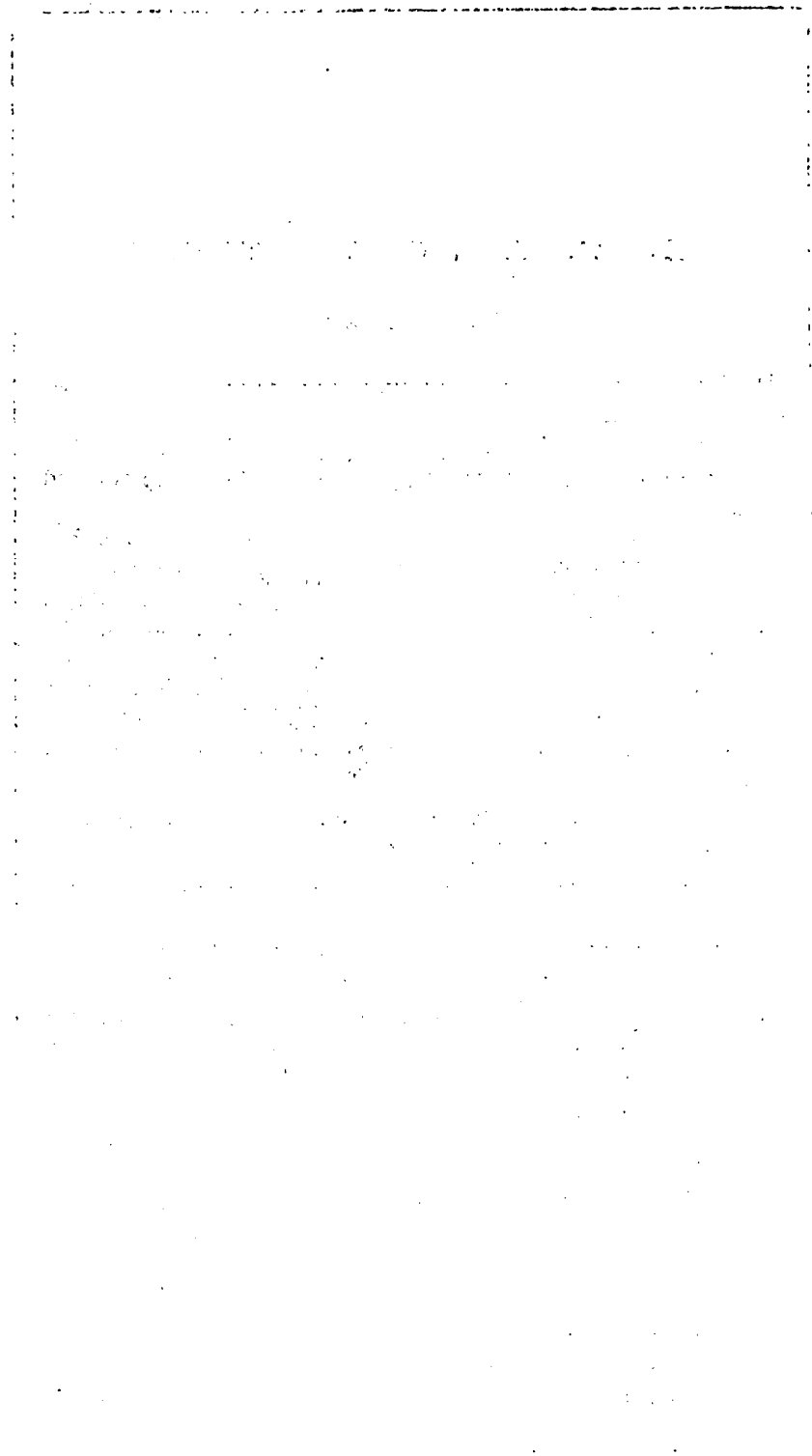
Wt. of mass of 1 gram	100 ^g	dynes (<i>g</i> in meters).	
Wt. of mass of 1 grain	6.47989 2 ^g	dynes (<i>g</i> in meters)	0.81156 8
1 foot-pound	13825.5 ^g	ergs (<i>g</i> in centimeters)	4.14068 2
1 kilogram-meter	100000 ^g	ergs (<i>g</i> in centimeters).	
1 watt	10 ⁷	ergs per sec.	
1 horse-power	76.0404 ^g	watts (<i>g</i> in meters)	1.88104 4
1 horse-power	746	watts (approximately)	2.87273 9

$$g = 32.086\ 528 + 0.171\ 293 \sin^2 \phi - 0.000\ 003\ h, \text{ in feet (Harkness).}$$

$$= 9.779\ 886 + 0.052\ 210 \sin^2 \phi - 0.000\ 003\ h, \text{ in meters (Harkness).}$$

$$l = 39.012\ 540 + 0.208\ 268 \sin^2 \phi - 0.000\ 000\ 3\ h, \text{ in inches (Harkness)}$$

$$= 0.990\ 910 + 0.005\ 290 \sin^2 \phi - 0.000\ 000\ 3\ h, \text{ in meters (Harkness).}$$



EXPLANATION OF THE TABLES.

INTRODUCTORY.

1. When we have a number with six or more decimal places, and we wish to use only five :

(a) If the sixth and following figures of the decimal are less than 5 in the sixth place, they are dropped ; thus, 0.46437 4999 is called 0.46437.

(b) If the sixth and following figures of the decimal are greater than 5 in the sixth place, the fifth place is increased by unity and the sixth and following places are dropped ; thus, 0.46437 5001 is called 0.46438.

(c) If the sixth figure of the decimal is 5, and if it is followed only by zeros, make the fifth figure the nearest *even* figure ; thus, 0.46437 500 is called 0.46438, while 0.46438 500 is also called 0.46438. The number is thus increased when the fifth figure is odd, and decreased when it is even, the two operations tending to neutralize each other in a series of computations, and hence to diminish the resultant error.

2. Hence any number obtained according to Art. 1 may be in error by half a unit in the fifth decimal place.

3. When the last figure of a number in these tables is 5, the number printed is too large, the 5 having been obtained according to Art. 1 (b) ; if the 5 is without the minus sign, the number printed is too small, the figures following the 5 having been dropped according to Art. 1 (a).

4. The marginal tables contain the products of the numbers at the top of the columns by 1, 2, 3, ... 9 *tenths*, and may be used in multiplying and dividing in interpolation.

(a) To multiply 38 by .746 :

$$\begin{aligned} 38 \times .7 &= &= 26.6 \\ 38 \times .4 &= 15.2 ; \therefore 38 \times .04 &= 1.52 \\ 38 \times .6 &= 22.8 ; \therefore 38 \times .006 &= .228 \\ \therefore 38 \times .746 &= &= 28.348 \end{aligned}$$

	38
1	3.8
2	7.6
3	11.4
4	15.2
5	19.0
6	22.8
7	26.6
8	30.4
9	34.2

multiplying by the second figure (hundredths), the decimal point table is moved *one* place to the left ; in multiplying by the third (ths), *two* to the left ; and so on.

(b) To divide 28 by 38 :

Dividend,	28				38
Next less,	26.6	corresponding to .7	1	3.8	
Remainder,	1 4		2	7.6	
Next less,	1 1.4	corresponding to .03	3	11.4	
Remainder,	2 6		4	15.2	
Nearest,	2 6.6	corresponding to .007	5	19.0	
			6	22.8	
			7	26.6	
			8	30.4	
			9	34.2	
∴ Quotient,		.737			

to the nearest third decimal place. The decimal point is moved one place to the right in each remainder, since the next figure in the quotient will be one place farther to the right.

To divide 23 by 38 :

Dividend,	23			
	22.8	corresponding to .6		
	2			
	0.0	corresponding to .00		
	2 0.			
Nearest,	1 9.0	corresponding to .005		
∴ Quotient,		.605		

The computer should use the marginal tables mentally.

LOGARITHMS.

5. The logarithm of a number is the exponent of the power to which a given number called the *base* must be raised to produce the first number. If $A = e^a$, a is called the logarithm of the number A to the base e , written $\log_e A = a$.

6. If $A = e^a$, and $B = e^b$, or (omitting subscripts) $\log A = a$, and $\log B = b$, we have

$$A \times B = e^{a+b}; \therefore \log(A \times B) = a + b; \therefore \log(A \times B) = \log A + \log B.$$

$$A \div B = e^{a-b}; \therefore \log(A \div B) = a - b; \therefore \log(A \div B) = \log A - \log B.$$

$$A^n = e^{na}; \therefore \log(A^n) = na; \therefore \log(A^n) = n \log A.$$

$$\sqrt[n]{A} = e^{\frac{1}{n}a}; \therefore \log \sqrt[n]{A} = \frac{1}{n}a; \therefore \log \sqrt[n]{A} = \frac{1}{n} \log A.$$

7. When the base is not specified, it is generally understood that logarithms to the base 10, or *common logarithms*, are meant. In this system, since

$$0.001 = \frac{1}{1000} = \frac{1}{10^3} = 10^{-3}, \quad \log 0.001 = -3;$$

$$0.01 = \frac{1}{100} = \frac{1}{10^2} = 10^{-2}, \quad \log 0.01 = -2;$$

$$0.1 = \frac{1}{10} = \frac{1}{10^1} = 10^{-1}, \quad \log 0.1 = -1;$$

$$1. = 10^0, \quad \log 1 = 0;$$

$$10. = 10^1, \quad \log 10 = +1;$$

$$100. = 10^2, \quad \log 100 = +2;$$

$$1000. = 10^3, \quad \log 1000 = +3.$$

8. The logarithm of a number between 100 and 1000 will be a number between 2 and 3, or $2 + m$ where m will be a decimal called the *mantissa*, the integral portion of the logarithm being the *characteristic*. The mantissa is always considered *positive*; thus $\log 0.002$ will be a number between -2 and -3 , that is, either $-3 + m$ or $-2 - m'$, the first form being used. We write $\log 0.002 = \bar{3}.30103$, the negative sign being placed over the characteristic to show that the characteristic alone is negative.

9. Since

$\log (A \times 10^n) = \log A + \log 10^n = \log A + n \log 10 = \log A + n$,
and $\log (A \div 10^n) = \log A - \log 10^n = \log A - n \log 10 = \log A - n$,
we have, if $\log 37.3 = 1.57171$,

$$\log 373. = 2.57171, \text{ and } \log 3.73 = 0.57171,$$

$$\log 3730 = 3.57171, \text{ and } \log 0.373 = \bar{1}.57171;$$

$$\log 37300 = 4.57171, \text{ and } \log 0.0373 = \bar{2}.57171.$$

Hence the position of the decimal point affects the characteristic alone, the mantissa being always the same for the same sequence of figures. For this reason the common system of logarithms is used in practice.

10. The characteristic is found as follows: *When the number is greater than 1, the characteristic is positive, and is one less than the number of digits to the left of the decimal point; when the number is less than 1, the characteristic is negative, and is one more than the number of zeros between the decimal point and the first significant figure.*

11. To avoid the use of negative characteristics we add 10 to the characteristic and write -10 after the mantissa, *i.e.* adding and subtracting the same quantity, 10. Thus $\log 0.2 = 1.30103$ would be written

9.30103 — 10. The — 10 is often omitted for brevity when there is no danger of confusion, but its existence must not be forgotten. Such logarithms are called *augmented* logarithms.

In this case the characteristic of the logarithm of a pure decimal is 9 diminished by the number of ciphers to the left of the first significant figure. Thus the characteristic of $\log 0.004$ is $9 - 2$, or 7, and that of $\log 0.94$ is $9 - 0$, or 9.

12. The arithmetical complement of the logarithm (written *colog*) of a number is the logarithm of its reciprocal, and is found by subtracting each figure of the logarithm from 9, commencing at the left, except the last significant figure on the right, which is subtracted from 10.

For $\log \frac{1}{x} = -\log x = 10 - \log x - 10$;
 thus, if $\log x = 2.46403$, $\text{colog } x = 7.53597 - 10$;
 if $\log x = 8.43000 - 10$, $\text{colog } x = 1.57000$.

TABLE I.

13. Page 3 contains the logarithms of numbers from 1 to 100, to five decimal places.

Pages 4–21 contain the mantissas of the logarithms of numbers from 1000 to 10009, to five decimal places.

Pages 22, 23, contain the mantissas of the logarithms of numbers from 10000 to 11009, to seven decimal places.

NOTE. — The mantissas of the logarithms of numbers, except those of the integral powers of 10, are incommensurable, the mantissas in the tables being found as shown in Art. 1.

To find the Logarithm of a Number.

14. The *characteristic* is found by the rules in Arts. 10 and 11, and the *mantissa* from the tables, as shown in Arts. 15, 16, 17, 18.

15. *When the number has four figures.* — Find on pages 4–21 the first three figures in the column marked *N*, and the fourth at the top of one of the other columns. The last three figures of the mantissa are found in this column on the horizontal line through the first three figures of the given number in column *N*. The first two figures of the mantissa are those under *L* in the same line with the number, or else those nearest above it, unless the last three figures of the mantissa as given in the tables are preceded by a *, when the first two figures are found under *L* in the first line *below* the number. Thus (page 4),

$\log 1136 = 3.05538$; $\log 1137 = 3.05576$; $\log 1138 = 3.05614$;
 $\log 1370 = 3.13672$; $\log 1371 = 3.13704$; $\log 1372 = 3.13735$;
 $\log 1380 = 3.13988$; $\log 1381 = 3.14019$; $\log 1382 = 3.14051$.

16. *When the number has less than four figures; annex ciphers on the right and proceed as in Art. 15. Thus,*

$$\log 1.13 = 0.05308; \log 12.8 = 1.10721; \log 130 = 2.11394;$$

$$\log 15 = 1.17609; \log 16 = 1.20412; \log 17 = 1.23045.$$

17. *When the number has more than four figures, as 11.4672. — Since the mantissa is independent of the position of the decimal point, point off the first four figures and find the mantissa of $\log 1146.72$. This will be between the mantissas of $\log 1146$ and $\log 1147$. Hence find from the tables the mantissas corresponding to 1146 and 1147; multiply the difference between them (called the tabular difference) by .72, and add the product (called the correction) to $\log 11.46$; the result will be the logarithm required.*

$$\text{Mantissa of } \log 1146 = 05918$$

$$\text{Mantissa of } \log 1147 = 05956$$

$$\text{Tabular difference} = 38$$

$$\log 11.46 = 1.05918$$

$$\text{correction} = 38 \times .72 = 27.36$$

$$\therefore \log 11.4672 = 1.05945 \text{ } 36$$

$$\text{or} = 1.05945$$

NOTE. — Since any mantissa in the tables may be in error by half a unit in the fifth decimal place (Art. 2), no advantage is gained by using the sixth place in the interpolated logarithm. Thus, according to Art. 1, we drop the .36, and call $\log 11.4672 = 1.05945$.

NOTE. — The marginal tables should be used in multiplying the tabular difference to find the correction (Art. 4).

NOTE. — It is assumed that the change in the mantissa is proportional to that in the number, as the latter increases from 1146 to 1147. An increase of 1 in the number causes an increase of 38 in the mantissa; hence an increase of .72 in the number will cause an increase of $38 \times .72$ in the mantissa.

NOTE. — We could also find the mantissa of $\log 11.4672$ by subtracting the product of the tabular difference by .28 (or $1.00 - .72$) from the mantissa corresponding to 1147; that is, the required mantissa is $05956 - (38 \times .28) = 05956 - 10.64 = 05945$ as before.

18. *The general rule is: Find the mantissa corresponding to the first four figures of the number; multiply the tabular difference by the fifth and following figures treated as a decimal; and add the product to the mantissa just found.*

The tabular difference is the difference between the mantissas corresponding to the two numbers in the tables, between which the given lies.

$$163 = 0.20995; \log 0.38024 = 1.58006; \log 0.0852763 = 2.93083;$$

$$89.524 = 2.27767; \log 0.38602 = 1.58661; \log 0.0085938 = 3.93419;$$

$$3.4 = 4.30067; \log 3.98743 = 0.60070; \log 0.090046 = 2.95446.$$

NOTE. — Page 3 is used when the number contains less than three figures, the number being found in the column *N*, and the logarithm in the column headed *Log*. The characteristic is given for whole numbers, and must be changed for decimals.

NOTE. — When a number is composed of three figures, find on pages 4–21 the number in the column *N*, and the mantissa corresponding in the column *L. o*.

To find the Number corresponding to a Given Logarithm.

19. From the tables we find the sequence of figures corresponding to the given mantissa, as shown in Arts. 20, 21, and 22, the position of the decimal point being determined by the characteristic (Arts. 10, 11).

20. *When the given mantissa can be found in the tables.*—Find on pages 4–21 the first two figures of the mantissa under *L* in the column headed *L. o*. The last three figures of the mantissa are then sought for in the columns headed 0, 1, 2, ... 9, in the same line with the first two figures, or in one of the lines just below, or in the line next above (where they would be preceded by a *). The first three figures of the required number will be found in the column headed *N*, in the same horizontal line with the last three figures of the mantissa, and the fourth figure of the number at the top of the column in which the last three figures of the mantissa are found. Thus (page 4),

$$\begin{aligned} 0.06221 &= \log 1.154; & 0.06558 &= \log 1.163; & 0.06893 &= \log 1.172; \\ 0.07004 &= \log 1.175; & 0.07188 &= \log 1.180; & 0.08063 &= \log 1.204. \end{aligned}$$

21. *When the given mantissa can not be found in the tables.*—If we wish to find the number whose logarithm is 2.16531, we enter the tables with 16531, and find that it lies between 16524 and 16554, so that the given mantissa corresponds to a number between 1463 and 1464. Also 16531 exceeds 16524 by 7, and this difference, divided by the tabular difference 30, gives .23... as the amount by which the required number exceeds 1463. Hence $2.16531 = \log 146.323...$, which we call 146.32, according to Art. 1, the incompleteness of the tables making the sixth figure uncertain.

NOTE. — The marginal tables should be used in dividing the difference between the given mantissa and the one next less in the tables by the tabular difference.

22. *The general rule is: Find the number corresponding to the mantissa in the tables next less than the given mantissa; divide the excess of the given mantissa over the one found in the tables by the tabular difference; and annex the quotient to the first four figures already found.*

The tabular difference is the difference between the two mantissas in the tables, between which the given mantissa lies.

$$\begin{aligned} 1.16600 &= \log 0.14656; & 0.18002 &= \log 1.5136; & 2.18200 &= \log 152.06; \\ 1.19000 &= \log 15.488; & 4.19680 &= \log 15773; & 1.20020 &= \log 15.856. \end{aligned}$$

23. For the use of the numbers S' , T' , S'' , T'' , see Arts. 35–38.

TABLE II.

24. This table (pages 26–70) contains the logarithms, to five decimal places, of the trigonometric sines, cosines, tangents, and cotangents of angles from 0° to 90° , for each minute. The logarithms in the columns headed *L. Sin*, *L. Tan*, and *L. Cos*, are augmented, and should be diminished by 10 (Art. 11), while those in the columns headed *L. Cot* are correctly given.

25. Since $\sec x = \frac{1}{\cos x}$, and $\operatorname{cosec} x = \frac{1}{\sin x}$, the logarithms of the secant and cosecant of an angle are the arithmetical complements of those of the cosine and sine respectively (Art. 12).

To find the Logarithmic Functions of an Angle Less than 90° .

26. *When the angle is less than 45°* , the degrees are found at the *top* of the page, and the minutes on the *left*. The numbers in the same horizontal line with the minutes of the angle are the logarithmic functions indicated by the notation at the *top* of the columns. Thus (page 34),

$$\begin{aligned} \log \sin 8^\circ 4' &= 9.14714 - 10, & \log \tan 8^\circ 4' &= 9.15145 - 10, \\ \log \cot 8^\circ 4' &= 0.84855, & \log \cos 8^\circ 4' &= 9.99568 - 10. \end{aligned}$$

27. *When the angle is greater than 45°* , the degrees are found at the *bottom* of the page, and the minutes on the *right*. The numbers in the same horizontal line with the minutes of the angle are the logarithmic functions indicated by the notation at the *bottom* of the columns. Thus (page 34),

$$\begin{aligned} \log \sin 81^\circ 25' &= 9.99511 - 10, & \log \tan 81^\circ 25' &= 0.82120, \\ \log \cot 81^\circ 25' &= 9.17880 - 10, & \log \cos 81^\circ 25' &= 9.17391 - 10. \end{aligned}$$

28. *When the angle is given to decimals of a minute.*—In finding $\log \sin 30^\circ 8'.48$, for example, we see that it will lie between the logarithmic sines of $30^\circ 8'$ and $30^\circ 9'$, that is, between 9.70072 and 9.70093, their difference 21 being the change in the logarithmic sine caused by a change of 1' in the angle. Hence, to find the correction to $\log \sin 30^\circ 8'$ that would give $\log \sin 30^\circ 8'.48$ we multiply 21 by .48 (Art. 4). The product 10.08 added to $\log \sin 30^\circ 8'$, since $\log \sin 30^\circ 9'$ is greater than $\log \sin 30^\circ 8'$, gives $\log \sin 30^\circ 8'.48 = 9.70082$ (Art. 1). Similarly, $\log \tan 30^\circ 8'.48 = 9.76391$, $\log \cot 30^\circ 8'.48 = 0.23609$, $\log \cos 30^\circ 8'.48 = 9.93691$, the correction being subtracted in the last two cases, since the cotangent and the cosine decrease as the angle increases.

29. *The general rule is: Find the function corresponding to the given degrees and minutes; multiply the tabular difference by the decimals of a minute; add the product to the function corresponding to the given degrees and minutes when finding the logarithmic sine or tangent, and subtract it when finding the logarithmic cosine or cotangent.*

The tabular differences are given in the columns headed $d.$ and $c. d.$, the latter containing the common difference for the $L. Tan$ and $L. Cot$ columns. The difference to be used is that between the functions corresponding to the two angles between which the given angle lies.

$$\begin{aligned} \text{For } 30^\circ 39'.38: \log \sin &= 9.70747; \log \cos = 9.93462; \\ \log \tan &= 9.77285; \log \cot = 0.22715. \end{aligned}$$

$$\begin{aligned} \text{For } 59^\circ 43'.46: \log \sin &= 9.93632; \log \cos = 9.70257; \\ \log \tan &= 0.23375; \log \cot = 9.76625. \end{aligned}$$

30. *When the angle is given to seconds, the correction may be found by multiplying the tabular difference by the number of seconds, and dividing the product by 60.*

To find the Acute Angle corresponding to a Given Logarithmic Function.

31. The column headed $L. Sin$ is marked $L. Cos$ at the bottom, while that headed $L. Cos$ is marked $L. Sin$ at the bottom; hence, if a logarithmic sine or cosine were given, we should expect to find it in one of these two columns. Similarly, we should expect to find a given logarithmic tangent or cotangent in one of the two columns headed $L. Tan$ and $L. Cot$.

32. *When the function can be found in the tables.*— If a logarithmic sine is given, find it in one of the two columns marked $L. Sin$ and $L. Cos$; if found in the column headed $L. Sin$, the degrees are taken from the top, and the minutes from the left of the page; if in the column headed $L. Cos$ but marked $L. Sin$ at the bottom, the degrees are taken from the bottom, and the minutes from the right of the page. Thus,

$$\begin{aligned} 9.70115 &= \log \sin 30^\circ 10'; & 9.93457 &= \log \sin 59^\circ 20'; \\ 9.93724 &= \log \cos 30^\circ 4'; & 9.70590 &= \log \cos 59^\circ 28'; \\ 9.76406 &= \log \tan 30^\circ 9'; & 0.23130 &= \log \tan 59^\circ 35'; \\ 0.23420 &= \log \cot 30^\circ 15'; & 9.76870 &= \log \cot 59^\circ 35'. \end{aligned}$$

33. *When the function can not be found in the tables.*— If we wish to find the angle whose logarithmic sine is 9.70170, we see on page 56 that the given logarithmic sine lies between 9.70159 and 9.70180, and

hence the angle is between $30^{\circ} 12'$ and $30^{\circ} 13'$. The given logarithmic sine differs from $\log \sin 30^{\circ} 12'$ by 11, and this difference, divided by the tabular difference 21, gives .52 + as the decimal of a minute by which the angle exceeds $30^{\circ} 12'$. Hence $9.70170 = \log \sin 30^{\circ} 12'.52$, which we call $30^{\circ} 12'.5$, since the incompleteness of the tables (Art. 1) makes the hundredths of a minute uncertain.

34. *The rule is: For a logarithmic sine or tangent find the degrees and minutes corresponding to the function in the tables next less than the given function; divide the difference between the given function and the one next less by the tabular difference; and the quotient will be the decimal of a minute to be added to the degrees and minutes already found. For a logarithmic cosine or cotangent find the degrees and minutes corresponding to the function next greater than the given function, since the cosine and cotangent decrease as the angle increases, and divide the difference between the given function and the one next greater by the tabular difference, to find the decimal of a minute.*

The tabular difference is the difference between the two functions in the tables, between which the given function lies.

$$9.70000 = \log \sin 30^{\circ} 4'.7; \quad 9.93500 = \log \sin 59^{\circ} 25'.7;$$

$$9.93400 = \log \cos 30^{\circ} 47'.6; \quad 9.70500 = \log \cos 59^{\circ} 32'.2;$$

$$9.77000 = \log \tan 30^{\circ} 29'.5; \quad 0.23200 = \log \tan 59^{\circ} 37'.4;$$

$$0.23300 = \log \cot 30^{\circ} 19'.1; \quad 9.76400 = \log \cot 59^{\circ} 51'.2.$$

Angles Near 0° or 90° .

35. The assumption that the variations in the functions are proportional to the variations in the angles if the latter are less than $1'$ fails when the angle is small, shown by the rapid changes in the tabular differences on pages 26, 27, and 28.

36. The quantities S' and T' which are used in this case are defined by the equations

$$S' = \log \frac{\sin \alpha}{\alpha'},$$

$$T' = \log \frac{\tan \alpha}{\alpha'},$$

where α' is the number of minutes in the angle. Their values from 0° to $1^{\circ} 40'$ ($=100'$) are given at the bottom of pages 3-21; from $1^{\circ} 40'$ to $3^{\circ} 20'$ at the left margin of pages 4 and 5, the first three being found at the top; and from 3° to 5° on page 24. Thus,

$$1' = 1' \text{ (page 5), } S' = 6.46 \ 373, \ T' = 6.46 \ 373;$$

$$15' = 15' \text{ (page 5), } S' = 6.46 \ 372, \ T' = 6.46 \ 373;$$

$$2^{\circ} 40' = 160' \text{ (page 5), } S' = 6.46 \ 357, \ T' = 6.46 \ 404;$$

$$4^{\circ} 20' = 260' \text{ (page 24), } S' = 6.46 \ 331, \ T' = 6.46 \ 456.$$

of these numbers should have -10 written after it (Art. 11).

NOTE. — The logarithmic cosine of a small angle is found by the ordinary method. The cotangent of an angle is the reciprocal of the tangent, and hence the logarithmic cotangent is the arithmetical complement of the logarithmic tangent. The formulas for finding the logarithmic cosine, tangent, and cotangent of angles near 90° are given on page 25.

37. *To find the logarithmic sine or tangent of a small angle.* — From Art. 36, we have

$$\log \sin \alpha = S' + \log \alpha',$$

$$\log \tan \alpha = T' + \log \alpha'.$$

Hence, to find the logarithmic sine or tangent of an angle less than 5° , find the value of the S' or T' corresponding to the angle, interpolating if necessary, and add it to the logarithm of the number of minutes in the angle.

Find $\log \sin 0^\circ 42'.6$. Since the angle is nearer $43'$ than $42'$, we take

$$S' = 6.46 \ 371$$

$$\log 42.6 = 1.62 \ 941$$

$$\therefore \log \sin 0^\circ 42'.6 = 8.09 \ 312$$

Find $\log \tan 1^\circ 53'.2$. Since the angle is nearer $1^\circ 53'$ ($= 113'$) than $114'$, we take

$$T' = 6.46 \ 388$$

$$\log 113.2 = 2.05 \ 385$$

$$\therefore \log \tan 1^\circ 53'.2 = 8.51 \ 773$$

NOTE. — When the angle is given in seconds, either reduce the seconds to decimals of a minute, or use the values of S'' and T'' given at the bottom of pages 3-23 and on page 24. They are defined by the equations

$$S'' = \log \frac{\sin \alpha}{\alpha''}, \text{ and } T'' = \log \frac{\tan \alpha}{\alpha''},$$

where α'' is the number of seconds in the angle. Hence

$$\log \sin \alpha = S'' + \log \alpha'', \text{ and } \log \tan \alpha = T'' + \log \alpha''.$$

38. *To find the small angle corresponding to a given logarithmic sine or tangent.* — From Art. 36,

$$\left. \begin{aligned} \log \alpha' &= \log \sin \alpha - S', \\ \log \alpha' &= \log \tan \alpha - T', \end{aligned} \right\}$$

or

$$\left. \begin{aligned} \log \alpha' &= \log \sin \alpha + \text{cpl } S', \\ \log \alpha' &= \log \tan \alpha + \text{cpl } T'. \end{aligned} \right\}$$

When the angle is less than 3° , find on pages 26-28 the value of $\text{cpl } S'$ (or $\text{cpl } T'$) corresponding to the function, interpolating if necessary, and add it to $\log \sin \alpha$ (or $\log \tan \alpha$); the sum will be the logarithm of the number of minutes in the angle.

In finding the angle whose logarithmic sine is 8.09006, we see from

the *L. Sin* column (page 26) that the angle is between $0^\circ 42'$ and $0^\circ 43'$, and that the value of $\text{cpl } S'$ must be either 3.53628 or 3.53629. The given logarithmic sine is nearer that of $42'$ than that of $43'$; hence we take

$$\begin{aligned}\text{cpl } S' &= 3.53628 \\ \log \sin \alpha &= 8.09006 \\ \log \alpha' &= 1.62634 \quad \therefore \alpha' = 42'.300.\end{aligned}$$

When the angle is between 3° and 5° , we may find S' and T' from page 24 after finding the angle approximately from pages 29 and 30. Thus in finding the angle whose logarithmic tangent is 8.77237 we find from page 29 that the angle is between $3^\circ 23'$ and $3^\circ 24'$, being nearer $3^\circ 23'$. Then on page 24 we have

$$\begin{aligned}T' &= 6.46423 \\ \log \tan \alpha &= 8.77237 \\ \therefore \log \tan \alpha - T' &= \log \alpha' = 2.30814 \quad \therefore \alpha' = 203'.30 = 3^\circ 23'.30.\end{aligned}$$

Angles Greater than 90° .

39. To find the logarithmic sine, cosine, tangent, or cotangent of an angle greater than 90° , subtract from the given angle the largest multiple of 90° contained therein. If this multiple is even, find from the tables the logarithmic sine, cosine, tangent, or cotangent of the remaining acute angle. If the multiple is odd, the logarithmic *cosine*, *sine*, *cotangent*, or *tangent*, respectively, of the remaining acute angle will be the function required; thus, $\sin 120^\circ = \sin (90^\circ + 30^\circ) = \cos 30^\circ$.

$x =$	I. QUADRANT. α	II. QUADRANT. $90^\circ + \alpha$	III. QUADRANT. $180^\circ + \alpha$	IV. QUADRANT. $270^\circ + \alpha$
$\sin x =$	$+\sin \alpha$	$+\cos \alpha$	$-\sin \alpha$	$-\cos \alpha$
$\cos x =$	$+\cos \alpha$	$-\sin \alpha$	$-\cos \alpha$	$+\sin \alpha$
$\tan x =$	$+\tan \alpha$	$-\cot \alpha$	$+\tan \alpha$	$-\cot \alpha$
$\cot x =$	$+\cot \alpha$	$-\tan \alpha$	$+\cot \alpha$	$-\tan \alpha$

Or we could find the difference between the angle and 180° or 360° , and find from the tables the same function of the remaining acute angle; thus, $\cos 300^\circ = \cos (360^\circ - 60^\circ) = \cos 60^\circ$, etc.

$\alpha =$	I. QUADRANT. α	II. QUADRANT. $180^\circ - \alpha$	III. QUADRANT. $180^\circ + \alpha$	IV. QUADRANT. $360^\circ - \alpha$ or $-\alpha$
$\sin x =$	$+\sin \alpha$	$+\sin \alpha$	$-\sin \alpha$	$-\sin \alpha$
$\cos x =$	$+\cos \alpha$	$-\cos \alpha$	$-\cos \alpha$	$+\cos \alpha$
$\tan x =$	$+\tan \alpha$	$-\tan \alpha$	$+\tan \alpha$	$-\tan \alpha$
$\cot x =$	$+\cot \alpha$	$-\cot \alpha$	$+\cot \alpha$	$-\cot \alpha$

To indicate that the trigonometric function is negative, n is written after its logarithm.

40. To find the angle corresponding to a given function, find the acute angle α corresponding thereto, and the required angle will be α , $180^\circ \pm \alpha$, or $360^\circ - \alpha$, according to the quadrant in which the angle should be placed.

41. There are always two angles less than 360° corresponding to any given function. Hence there will be ambiguity in the result unless some condition is known that will fix the angle; thus, if the sine is positive, the angle may be in either of the first two quadrants, but if we also know that the cosine is negative, the angle must be in the second quadrant.

Given One Function of an Angle, to find Another without finding the Angle.

42. Suppose $\log \tan \alpha = 9.79361$, and $\log \cos \alpha$ is sought. On page 57 the tabular difference for $\log \tan \alpha$ is 28, and that for $\log \cos \alpha$ is 8, the given logarithmic tangent exceeding 9.79354 by 7. Hence $28 : 7 = 8 : x$; $\therefore x = \frac{8}{28} \times 7 = 2 =$ correction to 9.92905, giving $\log \cos \alpha = 9.92903$.

In the margin are tables to facilitate the process. In the column headed $\frac{8}{28}$, the numerator is the tabular difference for the column headed *L. Cos*, and the denominator that for the logarithmic tangents. The numbers in the marginal column are the values of Δ , — the excess of $\log \tan \alpha$ over the next smaller logarithmic tangent, found in the tables, — such that $\frac{8}{28} \Delta$ shall be 0.5, 1.5, 2.5, etc.; and the numbers on the left are the corrections x to be applied to the numbers in the column headed *L. Cos*. Thus, if Δ is between 1.8 and 5.2, x is between 0.5 and 1.5, and is called 1. Note that 1 is opposite the *space* between 1.8 and 5.2. In the example above, the excess Δ is 7, which lies between 5.2 and 8.8; hence x is 2, the number on the left opposite the space between 5.2 and 8.8.

For example, if we have given the logarithms of the sides of a right-angled triangle, $\log a = 2.98227$ and $\log b = 2.90255$, to find the hypotenuse, we use the formulas

$$\tan \alpha = \frac{a}{b}, \text{ and } c = \frac{a}{\sin \alpha} = \frac{b}{\cos \alpha}.$$

$$\begin{aligned} \log a &= 2.98227 \quad (1) \\ \therefore \log \sin \alpha &= 9.88571 \quad (4) \\ \log b &= 2.90255 \quad (2) \\ \therefore \log \tan \alpha &= 0.07972 \quad (3) \\ \therefore \log c &= 3.09656 \quad (5) \end{aligned}$$

The value of $\log \tan \alpha$ being found in the column marked *L. Tan* at the bottom, the right column will contain the logarithmic sine of the corresponding angle. Also, the correction to 9.88563 is $20 \times \frac{1}{28}$, which we find to be 8 from the table headed $\frac{1}{28}$.

TABLE III.

43. This table (pages 72-94) contains the "natural" or actual numerical values of the trigonometric sines, cosines, tangents, and cotangents of angles from 0° to 90° , for each minute, while Table II. contains the logarithms of these numbers.

NOTE.—The secant is the reciprocal of the cosine, and the cosecant of the sine.

The arrangement and method of using the table are the same as those for Table II., except that the tabular differences are not printed. For the sake of clearness the first figures of the functions are given only opposite each fifth minute, and also when they change. Arts. 26, 27, 29, 30, 31, 32, and 34* will explain the table if the word "logarithmic" be changed to "natural," and "*L. Sin,*" etc., to "*N. Sin,*" etc.

$$\sin 20^\circ 10' = 0.34475 ; \cos 20^\circ 10' = 0.93869 ;$$

$$\tan 20^\circ 10' = 0.36727 ; \cot 20^\circ 10' = 2.7228.$$

$$\sin 68^\circ 10' = 0.92827 ; \cos 68^\circ 10' = 0.37191 ;$$

$$\tan 68^\circ 10' = 2.4960 ; \cot 68^\circ 10' = 0.40065.$$

In finding $\sin 68^\circ 24'.4$ we see that the required sine lies between 0.92978 and 0.92988, the tabular difference being 10; the correction for $0'.4$ is $10 \times .4 = 4$; hence $\sin 68^\circ 24'.4 = 0.92978 + 4$ units in the fifth place = 0.92982.

In finding $\cot 68^\circ 20'.4$ we see that the required cotangent lies between 0.39727 and 0.39694, the tabular difference being 33; the correction for $0'.4$ is $33 \times .4 = 13.2$; hence $\cot 68^\circ 20'.4 = 0.39727 - 13$ units in the fifth place = 0.39714.

NOTE.—The correction is added for the sine and tangent because they increase as the angle increases, and subtracted for the cosine and cotangent since they decrease as the angle increases.

In finding the angle whose tangent is 0.39870, the required angle will lie between $21^\circ 44'$ and $21^\circ 45'$, the tabular difference being $39896 - 39862 = 34$, while the given tangent exceeds that of $21^\circ 44'$ by $39870 - 39862 = 8$. Hence $8 \div 34$ or $0'.2+$ is the correction to be added to $21^\circ 44'$, giving $21^\circ 44'.2$ for the angle required.

In finding the angle whose cosine is 0.36850, the required angle will lie between $68^\circ 22'$ and $68^\circ 23'$, the tabular difference being $36867 - 36839 = 28$, while the given cosine is less than $\cos 68^\circ 22'$ by $36867 - 50$ or 17. Hence $17 \div 28$ or $0'.6+$ is the correction to be added giving $68^\circ 22'.6$ for the angle required.

* The examples in these articles apply only to Table II.

TABLE IV.

44. *Circular arcs with radius unity.* (Page 95.)—To find the length of the arc of $61^{\circ} 42' 35''.2$ in a circle whose radius is 200 feet, we find that in the circle whose radius is unity,

$$\text{Arc of } 61^{\circ} = 1.06465\ 08$$

$$\text{Arc of } 42' = 0.01221\ 73$$

$$\text{Arc of } 35'' = 0.00016\ 97$$

$$\text{Arc of } 0''.2 = 0.00000\ 10$$

$\therefore$ Arc of $61^{\circ} 42' 35''.2 = 1.07703\ 88^*$ feet in the circle whose radius is 1 foot, and if the radius is 200 feet the length of the arc will be $1.07703\ 88 \times 200$.

To find the angle at the center of a circle with radius 3, the length of its intercepted arc being 13.39410 00: the length of its intercepted arc in the circle whose radius is unity will be $\frac{1}{3} \times 13.39410\ 00 = 4.46470\ 00$.

$$4.46470\ 00$$

$$\text{Next less} = 3.14159\ 27 \quad \text{corresponding to } 180^{\circ}.$$

$$\text{Difference} = 1.32310\ 73$$

$$\text{Next less} = 1.30899\ 69 \quad \text{corresponding to } 75^{\circ}.$$

$$.01411\ 04$$

$$.01396\ 26 \quad \text{corresponding to } 48'.$$

$$.00014\ 78$$

$$.00014\ 54 \quad \text{corresponding to } 30''.$$

$$.00000\ 24 \quad \text{corresponding to } 0''.5.$$

$$\therefore 255^{\circ} 48' 30''.5.$$

* Owing to the incompleteness of the table the last figure will probably be erroneous.

TABLE V.

45. *Conversion of common logarithms into Naperian, and vice versa* (page 96).—We have

$$\log_{10} N = M \log_e N, \text{ and } \log_e N = \frac{1}{M} \log_{10} N.$$

Table V. contains the multiples of M and $\frac{1}{M}$ by numbers from 1 to 100.

Find the common logarithm of 2, its Naperian logarithm being 0.69314 718056.

$$\begin{aligned} M \times .69 &= 0.29966 \ 31925 \\ M \times .0031 &= .00134 \ 63128 \ 94 \\ M \times .000047 &= .00002 \ 04118 \ 41 \\ M \times .00000018 &= .00000 \ 00781 \ 73 \\ M \times .0000000005 &= .00000 \ 00002 \ 17 \\ M \times .00000000006 &= .00000 \ 00000 \ 26 \\ \therefore \log_{10} 2 &= 0.30102 \ 99956 \ 51 \end{aligned}$$

(True value = 0.30102 99957)

Find the Naperian logarithm of 0.2, its common logarithm being 9.30102 99957 - 10. Hence the true logarithm is

$$\log_{10} 0.2 = -1 + .30102 \ 99957 = -0.69897 \ 00043.$$

$$\begin{aligned} \frac{1}{M} \times .69 &= 1.58878 \ 37142 \\ \frac{1}{M} \times .0089 &= .02049 \ 30073 \ 28 \\ \frac{1}{M} \times .000070 &= .00016 \ 11809 \ 57 \\ \frac{1}{M} \times .0000000043 &= .00000 \ 00099 \ 01 \\ \therefore \log_e 0.2 &= -1.60943 \ 79123 \ 86 \end{aligned}$$

(True value = -1.60943 79124)



